[3]B. d'Espagnat and M. K. Gaillard, Phys. Letters 25B, 346 (1967).

[4]S. Fubini and G. Furlan, Physics 1, 229 (1965). See also G. Furlan, F. G. Lannoy, C. Rossetti, and G. Segrè, Nuovo Cimento 38, 1747 (1965).

[5]See the discussion in L. B. Auerbach, J. M. Dobbs, A. K. Mann, W. K. McFarlane, D. H. White, R. Cester, P. T. Eschstruth, G. K. O'Neill, and D. Yount, Phys. Rev. 155, 1505 (1967).

[6]C. G. Callan and S. B. Treiman, Phys. Rev. Letters 16, 153 (1966).

[7]Relaxation of the assumption that $m_{0,2;n} \approx m_K$ and hence that $t_n^{(0),(2)}(t) \approx t$ introduces, in general, the unknown parameters $m_{0,2;n}$ into $g_n^{(0),(2)}(t)$ and $\beta_n^{(0),(2)}(t)$. However, since it is necessary only to cover the range from 0 to 0.73 for $[g^{(2)}(t)]^2 - [g^{(0)}(t)]^2$, a linear parametrization for $g^{(0),(2)}(t)$ vs $t$ with a small slope should still be reasonable, even though this slope would no longer be simply $\langle m_{\mp 1;n}\rangle^{-2}$ but would also depend on $\langle m_{0,2;n}\rangle/m_K$. Our result for $f_-(t)$ would nevertheless remain unchanged because, as seen from Eqs. (14) and (15), $f_-(t)$ does not depend on the slope.

---

# CROSS-SECTION SUM RULES FOR NEUTRINO (ANTINEUTRINO) SCATTERING BY NUCLEI FROM CURRENT COMMUTATION RELATIONS*

C. W. Kim
Department of Physics, The Johns Hopkins University, Baltimore, Maryland

and

Michael Ram
Department of Physics, The State University of New York, Buffalo, New York

(Received 4 October 1967)

Total differential cross-section sum rules for forward neutrino (antineutrino) scattering by nuclei are derived using the weak charge commutation relations conjectured by Gell-Mann on the basis of the quark model, and the closure approximation. These sum rules can now be tested with neutrino (antineutrino) beams available at Brookhaven National Laboratory and CERN.

In this Letter, we wish to apply the method recently developed[1] in the calculation of the total muon capture rate in $He^3$ to derive sum rules for the total differential cross section for forward scattering of neutrinos (antineutrinos) by nuclei. In applying the method we make use of the modifications suggested by Primakoff.[2] The sum rules we derive are the following:

$$\frac{d\sigma^{(\nu)}(A,I,-I_3;\theta=0)}{d\cos\theta} - \frac{d\sigma^{(\nu)}(A,I,I_3;\theta=0)}{d\cos\theta} \cong 8I_3\frac{G^2\cos\theta_C^{\,2}\nu^2}{\pi} \quad (I_3>0); \tag{1a}$$

$$\frac{d\sigma^{(\bar\nu)}(A,I,I_3;\theta=0)}{d\cos\theta} - \frac{d\sigma^{(\bar\nu)}(A,I,-I_3;\theta=0)}{d\cos\theta} \cong 8I_3\frac{G^2\cos\theta_C^{\,2}\nu^2}{\pi} \quad (I_3>0); \tag{1b}$$

where $G=(1.02/m_p^2)\times 10^{-5}$ and $\cos\theta_C = 0.98$. In the above, $d\sigma^{(\nu \text{ or } \bar\nu)}(A,I,I_3;\theta)/d\cos\theta$ is the total $\Delta S=0$ differential cross section for the reaction $(\nu_l \text{ or } \bar\nu_l)+N_a \to (l \text{ or } \bar l)+N_b$. In our notation, $N_a = (A,I,I_3)$ is the $I_3$ member ($I_3$ is the third component of isospin) of a nuclear isospin multiplet of isospin $I$ and mass number $A$, and $N_b$ is any allowed final state of hadrons with zero strangeness. The angle $\theta$ is the angle of the final lepton $l=e$ or $\mu$ relative to the original neutrino (antineutrino) direction, $\nu$ is energy of the incident neutrino (antineutrino), and $m_p$ is the proton mass.

We now briefly sketch the essential parts of the derivation, leaving details for a future paper. Using the current-current–type Hamiltonian for weak interactions, one can show that

$$\frac{d\sigma^{(\nu)}(N_a;\theta)}{d\cos\theta} = \sum_b \frac{d\sigma^{(\nu)}(N_a\to N_b;\theta)}{d\cos\theta} = \frac{(G\cos\theta_C)^2}{2\pi}\frac{1}{(2J_a+1)}\sum_b l^2\frac{(E-l)}{(E-V\cos\theta)}\mathfrak{L}_{\alpha\beta}^{(\nu)}\mathfrak{M}_{\alpha\beta}^{(\nu)}(N_a\to N_b). \tag{2}$$

We are assuming that the lepton mass can be neglected so that its four-momentum is simply $p_l \cong (\vec{l}, il)$.

In the above,

$$|\vec{l}| = l \cong \frac{(M_a{}^2 - M_b{}^2) + 2\nu M_a}{2(E - \nu\cos\theta)}, \tag{3}$$

$$\mathcal{L}_{\alpha\beta}{}^{(\nu)} = (\nu l)^{-1}\{(p_\nu)_\alpha (p_l)_\beta + (p_l)_\alpha (p_\nu)_\beta - (p_\nu \cdot p_l)\delta_{\alpha\beta} + \epsilon_{\alpha\beta\rho\sigma}(p_\nu)_\rho (p_l)_\sigma\}, \tag{4}$$

$$\mathfrak{N}_{\alpha\beta}{}^{(\nu)}(N_a \to N_b) = \sum_{M_a=-J_a}^{J_a} \sum_{M_b=-J_b}^{J_b} \Omega^2 \langle N_a; \vec{p}_a = 0 | \mathcal{J}_\beta{}^{(-)}(0) | N_b; \vec{p}_b\rangle\langle N_b; \vec{p}_b | \mathcal{J}_\alpha{}^{(+)}(0) | N_a; \vec{p}_a = 0\rangle, \tag{5}$$

$$\mathcal{J}_\alpha{}^{(\pm)}(x) = V_\alpha{}^{(\pm)}(x) + A_\alpha{}^{(\pm)}(x), \tag{6}$$

and

$$p_\nu = (\vec{\nu}, i\nu); \quad E = M_a + \nu.$$

The states $|N_a\rangle$ and $|N_b\rangle$ have spins $J_a$ and $J_b$ and masses $m_a$ and $m_b$, respectively. The summations in Eq. (5) extend over the third components of the nuclear spins $M_a$ and $M_b$. Finally, $V_\alpha{}^{(\pm)}(x)$ and $A_\alpha{}^{(\pm)}(x)$ are the vector and axial-vector, strangeness-conserving, weak hadron currents, respectively. We have assumed the initial nucleus is at rest and set $\vec{p}_a = 0$.

Introducing[3]

$$Q_\alpha{}^{(\pm)}(m_b; \vec{l}, \vec{\nu}) \equiv \int \mathcal{J}_\alpha{}^{(\pm)}(\vec{x}, t=0) \exp[\pm i\{\vec{\nu} - l(m_b)\hat{l}\} \cdot \vec{x}] d\vec{x}, \tag{7}$$

where $\hat{l} = \vec{l}/l$, and using the closure approximation,[4] one can show that

$$\frac{1}{K(N_a;\theta)} \frac{d\sigma^{(\nu)}(N_a;\theta)}{d\cos\theta} \cong \frac{(G\cos\theta_C)^2}{2\pi} \frac{1}{(2J_a+1)} \sum_{M_a=-J_a}^{J_a} \mathcal{L}_{\alpha\beta}{}^{(\nu)} \mathfrak{N}_{\alpha\beta}{}^{(\nu)\prime}(N_a). \tag{8}$$

In the above,

$$K(N_a;\theta) = \langle l^2(E-l)\rangle_{\text{av}}/(E - \nu\cos\theta), \tag{9}$$

and

$$\mathfrak{N}_{\alpha\beta}{}^{(\nu)\prime}(N_a) = \langle N_a; \vec{p}_a = 0 | Q_\beta{}^{(-)}(\langle m_b\rangle_a; \vec{l}, \vec{\nu}) Q_\alpha{}^{(+)}(\langle m_b\rangle_a; \vec{l}, \vec{\nu}) | N_a; \vec{p}_a = 0\rangle. \tag{10}$$

The notation $\langle\cdots\rangle_a$ in Eqs. (9) and (10) denotes a suitable average over all possible final states $|N_b\rangle$. Consider the isospin doublet $(|He^3\rangle, |H^3\rangle)$. We have

$$\mathfrak{N}_{\alpha\beta}{}^{(\nu)\prime}(H^3) = \langle H^3; \vec{p}_{H^3} = 0 | Q_\beta{}^{(-)}(\langle m_b\rangle_{H^3}; \vec{l}, \vec{\nu}) Q_\alpha{}^{(+)}(\langle m_b\rangle_{H^3}; \vec{l}, \vec{\nu}) | H^3; \vec{p}_{H^3} = 0\rangle. \tag{11}$$

Using the relations

$$e^{i\pi I_1}|He^3; \vec{p}_{He^3} = 0\rangle = |H^3; \vec{p}_{H^3} = 0\rangle, \quad e^{i\pi I_1}\mathcal{J}_\alpha{}^{(\pm)} e^{-i\pi I_1} = \mathcal{J}_\alpha{}^{(\mp)}, \tag{12}$$

where $I_1$ is the first component of isospin, one can easily show that

$$\mathfrak{N}_{\alpha\beta}{}^{(\nu)\prime}(He^3) = \langle H^3; \vec{p}_{H^3} = 0 | Q_\beta{}^{(+)}(\langle m_b\rangle_{H^3}; \vec{l}, \vec{\nu}) Q_\alpha{}^{(-)}(\langle m_b\rangle_{H^3}; \vec{l}, \vec{\nu}) | H^3; \vec{p}_{H^3} = 0\rangle + \Delta_{\alpha\beta}{}^{(\nu)}, \tag{13}$$

where[5]

$$\Delta_{\alpha\beta}^{(\nu)}=\langle H^3;\vec{p}_{H^3}=0|Q_\beta^{(+)}(\langle m_b\rangle_{He^3};-\vec{l}-\vec{\nu})Q_\alpha^{(-)}(\langle m_b\rangle_{He^3};-\vec{l},-\vec{\nu})|H^3;\vec{p}_{H^3}=0\rangle$$
$$-\langle H^3;\vec{p}_{H^3}=0|Q_\beta^{(+)}(\langle m_b\rangle_{H^3};\vec{l},\vec{\nu})Q_\alpha^{(-)}(\langle m_b\rangle_{H^3};\vec{l},\vec{\nu})|H^3;\vec{p}_{H^3}=0\rangle. \quad (14)$$

Combining Eqs. (8), (11), and (13) we find that

$$\frac{1}{K(H^3;\theta)}\frac{d\sigma^{(\nu)}(H^3;\theta)}{d\cos}-\frac{1}{K(He^3;\theta)}\frac{d\sigma^{(\nu)}(He^3;\theta)}{d\cos\theta}$$
$$\cong-\frac{(G\cos\theta_C)^2}{2\pi}\frac{1}{2}\sum_{M_{H^3}=-\frac{1}{2}}^{\frac{1}{2}}\langle H^3;\vec{p}_{H^3}=0|\mathcal{S}_{\alpha\beta}[Q_\alpha^{(+)}(\langle m_b\rangle_{H^3};\vec{l},\vec{\nu}),Q_\beta^{(-)}(\langle m_b\rangle_{H^3};\vec{l},\vec{\nu})]$$
$$-\mathcal{a}_{\alpha\beta}\{Q_\alpha^{(+)}(\langle m_b\rangle_{H^3};\vec{l},\vec{\nu}),Q_\beta^{(-)}(\langle m_b\rangle_{H^3};\vec{l},\vec{\nu})\}|H^3;\vec{p}_{H^3}=0\rangle-\Delta^{(\nu)}, \quad (15)$$

where

$$\mathcal{S}_{\alpha\beta}=(\nu l)^{-1}\{(p_\nu)_\alpha(p_l)_\beta+(p_l)_\alpha(p_\nu)_\beta-(p_\nu\cdot p_l)\delta_{\alpha\beta}\}, \quad (16a)$$

$$\mathcal{a}_{\alpha\beta}=(\nu l)^{-1}\epsilon_{\alpha\beta\rho\sigma}(p_\nu)_\rho(p_l)_\sigma, \quad (16b)$$

$$\Delta^{(\nu)}=\frac{(G\cos\theta_C)^2}{2\pi}\frac{1}{2}\sum_{M_{H^3}=-\frac{1}{2}}^{\frac{1}{2}}\mathcal{L}_{\alpha\beta}^{(\nu)}\Delta_{\alpha\beta}^{(\nu)}. \quad (17)$$

If we restrict ourselves to forward scattering, i.e., $\theta=0$, Eq. (15) reduces to

$$\frac{d\sigma^{(\nu)}(H^3;\theta=0)}{d\cos\theta}-\frac{d\sigma^{(\nu)}(He^3;\theta=0)}{d\cos\theta}\cong-\frac{(G\cos\theta_C)^2}{2\pi}\nu^2\frac{1}{2}\sum_{M_{H^3}=-\frac{1}{2}}^{\frac{1}{2}}\langle H^3;\vec{p}_{H^3}=0|\mathcal{S}_{\alpha\beta}$$
$$\times[Q_\alpha^{(+)}(\langle m_b\rangle_{H^3};\vec{l}\cong\vec{\nu},\vec{\nu}),Q_\beta^{(-)}(\langle m_b\rangle_{H^3};\vec{l}\cong\vec{\nu},\vec{\nu})]|H^3;\vec{p}_{H^3}=0\rangle. \quad (18)$$

In writing relation (18), we have assumed in the spirit of the closure approximation[4] that $\langle m_b\rangle_{H^3 \text{ or } He^3}-m_{H^3}\ll\nu$ and $\langle m_b\rangle_{H^3 \text{ or } He^3}-m_{H^3}\ll m_{H^3}$.[6] Using the equal-time commutation relations for the integrated weak current $J_\alpha^{(\pm)}$ conjectured by Gell-Mann[7] in the context of the quark model, Eq. (18) becomes

$$\frac{d\sigma^{(\nu)}(H^3;\theta=0)}{d\cos\theta}-\frac{d\sigma^{(\nu)}(He^3;\theta=0)}{d\cos\theta}\cong-\frac{(G\cos\theta_C)^2}{\pi}\nu^2\{8I_3(H^3)\}=\frac{4(G\cos\theta_C)^2}{\pi}\nu^2. \quad (19)$$

Relation (19) can be immediately generalized to apply to the case of an initial nucleus of arbitrary isospin $I$, so that Eq. (1a) follows. It is easy to show that the corresponding sum rule for antineutrino scattering is similar to that for neutrino scattering as indicated in Eq. (1b).

We wish to make the following remarks:

(i) Nonzero contributions to sum rules come only from the equal-time commutators

$$[\int V_j^{(+)}(\vec{x},0)d\vec{x},\int V_j^{(-)}(\vec{y},0)d\vec{y}]=[\int A_j^{(+)}(\vec{x},0)d\vec{x},\int A_j^{(-)}(\vec{y},0)d\vec{y}]=2I_3, \quad (20)$$

where $j=0, 1, 2, 3$, $V_0^{(\pm)}=-iV_4^{(\pm)}$, and $A_0^{(\pm)}=-iA_4^{(\pm)}$.

(ii) The right-hand side of Eq. (1) depends on only one nuclear parameter, namely $I_3$.

(iii) Relations (1) apply to both $\nu_e$ and $\nu_\mu$ scattering with an electron and a muon in the final state, respectively. Nevertheless, since we have made the approximation $m_l^2 \ll l^2$, and the assumption $\langle m_b\rangle_a - m_a \ll \nu$, the domain of validity will be different in the two cases. For $\nu_e$ scattering, we expect the sum rules to be valid to a good approximation for neutrino energies of 100 MeV or more. Such $\nu_e$ beams of reasonable intensity are not available at present. In the case of $\nu_\mu$ scattering, the sum rules should be valid, e.g., for incident $\nu_\mu$ energies of 1 BeV or more. Beams of $\nu_\mu$ of reasonable intensity and appropriate energy are already available at Brookhaven and CERN, so that tests of Eq. (1) for $\nu_\mu$ scattering are already feasible. This is one of the advantages of sum rules (1) over the other similar[8] sum rules that have been derived and can only be tested at energies of at least 5 BeV.[9] The second advantage is that sum rules (1) apply to $\nu$ scattering by complex nuclei, which corresponds to the usual experimental situation, while the sum rules derived in Ref. 8 concern neutrino (antineutrino) scattering by nucleons.

(iv) In only a few cases such as that of the doublet ($|He^3\rangle$, $|H^3\rangle$) are both nuclei appearing in sum rules (1) relatively stable, and experimental tests of the relations possible. Nevertheless, in many cases only one of the nuclei will be stable, and relations (1) can then provide a lower limit for one of the cross sections.

(v) Goulard and Primakoff[10] calculated the "elastic" scattering cross sections $d\sigma^{(\nu)}(A;\theta)/d\cos\theta$ and $d\sigma^{(\nu)}(A;\theta)/d\cos\theta$ using the impulse and closure approximations. For the cases in which we could compare results, e.g., for ($|He^3\rangle$, $|H^3\rangle$) and ($|Si^{27}\rangle$, $|Al^{27}\rangle$), our cross sections are larger than theirs by a factor of about 1.7. This difference is due to the fact that, by using the current commutation relations, meson-exchange corrections as well as "inelastic" scattering corrections (e.g., pion production) are included in our results, while such corrections are neglected in Ref. 10.

(vi) The cross sections in Eq. (1) do not include $\Delta S=1$ processes. The inclusion of these transitions is not expected to change the cross section by much more than 4 % because of the factor $\sin^2\theta = 0.04$ involved in the strangeness-changing reactions.

One of the authors (C.W.K.) would like to thank Professor H. Primakoff for discussions. He would also like to thank the Institute for Theoretical Physics, State University of New York at Stony Brook, for its hospitality, where a part of this work was done. The other author (M.R.) would like to thank Professor D. Lin for discussions.

---

*Work supported in part by the National Science Foundation.

[1]C. W. Kim and Michael Ram, Phys. Rev. Letters 18, 508 (1967).

[2]H. Primakoff, in High Energy Physics and Nuclear Structure, edited by G. Alexander (North-Holland Publishing Co., Amsterdam, The Netherlands, 1967), pp. 409-445.

[3]In Eq. (7) we have written $l(m_b)$ in order to remind the reader of the explicit dependence of $l$ on $m_b$.

[4]H. Primakoff, Rev. Mod. Phys. 31, 802 (1959).

[5]The quantities $\langle m_b^2\rangle_{H^3}$ and $\langle m_b^2\rangle_{He^3}$ are slightly different since for the first the final states that contribute are $|N_b\rangle = |He^3\rangle, |H^2,p\rangle, |ppn\rangle, \cdots$, while for the second they are $|N_b\rangle = |ppp\rangle, |ppp\pi^0\rangle, |ppn\pi^+\rangle, \cdots$.

[6]It is easy to see from Eq. (3) that in the forward direction

$$1-l(\langle M_b\rangle_{H^3 \text{ or } He^3})/\nu \cong [\langle M_b^2\rangle_{H^3 \text{ or } He^3} - M_{H^3}^2]/2\nu M_{H^3} \ll 1,$$

so that

$$\Delta^{(\nu)} \sim [\langle m_b^2\rangle_{H^3 \text{ or } He^3} - M_{H^3}^2]/2\nu M_{H^3} \ll 1$$

and can be neglected.

[7]M. Gell-Mann, Phys. Rev. 125, 1067 (1962), and Physics 1, 63 (1964).

[8]S. L. Adler, Phys. Rev. 143, 1144 (1966).

[9]S. L. Adler and F. J. Gilman, Phys. Rev. 156, 1598 (1967).

[10]B. Goulard and H. Primakoff, Phys. Rev. 135, B1139 (1964).