CONTRIBUTION OF "LEAKAGE" MATRIX ELEMENTS TO THE FORM FACTORS IN K_{l3} DECAY*

A. K. Mann and H. Primakoff
Physics Department, University of Pennsylvania, Philadelphia, Pennsylvania 19104
(Received 29 November 1967)

A theoretical expression for the form factor $f_-(t)$ in K_{l3} decay is obtained from a sum rule based on the hadron weak-current commutation relations. The "leakage" matrix elements are found to make an important contribution to $f_-(t)$.

Recent experimental studies of the K_{l3} decays lead to the following conclusions[1]:

(1) No important discrepancies exist with respect to the predictions of the leptonic $\Delta I=\frac{1}{2}$ rule; in particular, the form factors $f_\pm(t)=f_\pm(-[p_K-p_\pi]^2)$ are essentially the same for $K^+\to\pi^0+l^++\nu$ and $K_L{}^0\to\pi^\pm+l^\mp+\nu$ over the entire physical region of t: $m_l{}^2\leqslant t\leqslant(m_K-m_\pi)^2$.

(2) With the definitions $\lambda_\pm=[m_K{}^2/f_\pm(0)][df_\pm(t)/dt]_{t=0}$, $\lambda_\pm'=[m_K{}^4/2f_\pm(0)][d^2f_\pm(t)/dt^2]_{t=0},\cdots$, one obtains from the observed pion momentum spectrum in K_{e3} the values $\lambda_+=0.25\pm0.08$, $\lambda_+'\simeq0$, $\lambda_+''\simeq0,\cdots$.

(3) Assuming $\lambda_-,\lambda_-',\cdots$ to be not too large, the values of $\xi\equiv f_-(0)/f_+(0)$ calculated on the one hand from the observed values of the $K_{\mu3}$ muon polarization, and on the other, from the observed values of the $K_{\mu3}/K_{e3}$ branching ratio, are mutually inconsistent, viz., $\xi_{\rm pol}=-1.0\pm0.2$ and $\xi_{\rm br}=0.6\pm0.3$; moreover, the uncertainties in the measurements are such that no trustworthy values of $\lambda_-,\lambda_-',\cdots$ have yet emerged.

The experimental values of λ_+, $\lambda_+',\cdots$, ξ, and $\lambda_-,\lambda_-',\cdots$ may be compared with available theoretical values. Thus, for example, the K^*, κ pole model for $f_\pm(t)$ predicts[2] $\lambda_+=0.32$, $\lambda_+'=0.10$, $\xi=0.15$, $\lambda_-=0.78$, and $\lambda_-'=0.46$ for $m(K^*)=890$ MeV and $m(\kappa)=730$ MeV.

To throw further light on the values of $\xi,\lambda_-,\lambda_-',\cdots$ from a theoretical point of view, we develop a procedure based on a recent elegant paper by d'Espagnat and Gaillard,[3] who applied the Fubini-Furlan[4] sum rule to the calculation of $f_-(t)$. The essential feature of the d'Espagnat-Gaillard calculation is to make explicit a t dependence of the $K\to\pi$ matrix elements of $\partial_\lambda V_\lambda{}^\dagger$ and $\partial_\lambda V_\lambda$, the divergences of the hadronic $\Delta S=\Delta Q$ polar-vector weak currents, which has hitherto not been pointed out. This t dependence arises in part kinematically from the mass difference of the pion and kaon and in part dynamically from the so-called "leakage" matrix elements of $\partial_\lambda V_\lambda{}^\dagger$ and $\partial_\lambda V_\lambda$ which connect a given pion state to states outside the $\{\pi,\eta,K\}$ pseudoscalar octet.

We begin our discussion with the hadron weak-current commutation relation

$$\langle\pi^+|\int V_0{}^\dagger(\vec{x},0)d\vec{x},\int V_0(\vec{y},0)d\vec{y}|\pi^+\rangle=\langle\pi^+|Q+Y|\pi^+\rangle=1. \tag{1}$$

Note that the contributing kaon state is $|\bar{K}^0\rangle$, and label the contributing states outside the $\{\pi,\eta,K\}$ octet as $\bar{X}_n{}^{(0)}(Q=0,S=-1)$ and $X_n{}^{(2)}(Q=2,S=+1)$. The Fubini-Furlan (FF) sum rule then follows immediately from Eq. (1), viz.,

$$\beta^{(K)}(t)[f(t)]^2+\sum_n\beta_n{}^{(0)}(t)[g_n{}^{(0)}(t)]^2-\sum_n\beta_n{}^{(2)}(t)[g_n{}^{(2)}(t)]^2=1, \tag{2}$$

where

$$4E_K(t)E_\pi(t)(m_K{}^2-m_\pi{}^2)^{-2}|\langle\pi^+|\partial_\lambda V_\lambda{}^\dagger|\bar{K}^0\rangle|^2\equiv[f(t)]^2=\left[f_+(t)+\frac{t}{m_K{}^2-m_\pi{}^2}f_-(t)\right]^2,$$

$$E_K(t)\mp E_\pi(t)=[|\vec{p}(t)|^2+m_K{}^2]^{1/2}\mp[|\vec{p}(t)|^2+m_\pi{}^2)]^{1/2}=(m_K{}^2-m_\pi{}^2)^{1/2}\left[\frac{t}{m_K{}^2-m_\pi{}^2}\right]^{\pm1/2},$$

$$\beta^{(K)}(t)=\frac{(m_K{}^2-m_\pi{}^2)^2}{4E_K(t)E_\pi(t)[E_K(t)-E_\pi(t)]^2}=\left[1-\frac{t^2}{(m_K{}^2-m_\pi{}^2)^2}\right]^{-1}, \tag{3}$$

and

$$4E_n^{(0)}(t)E_\pi(t)(m_{0;n}^2-m_\pi^2)^{-2}|\langle\pi^+|\partial_\lambda V_\lambda^\dagger|\overline{X}_n^{(0)}\rangle|^2\equiv[g_n^{(0)}(t)]^2,$$

$$E_n^{(0)}(t)-E_\pi(t)=[|\vec{p}(t)|^2+m_{0;n}^2]^{1/2}-[|\vec{p}(t)|^2+m_\pi^2]^{1/2}\equiv[t_n^{(0)}(t)]^{1/2},$$

$$\beta_n^{(0)}(t)=\frac{(m_{0;n}^2-m_\pi^2)^2}{4E_n^{(0)}(t)E_\pi(t)[E_n^{(0)}(t)-E_\pi(t)]^2}=\left[1-\frac{[t_n^{(0)}(t)]^2}{(m_{0;n}^2-m_\pi^2)^2}\right]^{-1},\tag{4}$$

with $m_{0;n}\geq m_K+m_\pi$ and a sum implied over spin orientations of $|\overline{X}_n^{(0)}\rangle$ in the definition of the "leakage" form factor $g_n^{(0)}(t)$ if this spin is >0; analogous expressions for $g_n^{(2)}(t)$ and $\beta_n^{(2)}(t)$ can be obtained by replacing $\partial_\lambda V_\lambda^\dagger$, $|\overline{X}_n^{(0)}\rangle$, and $m_{0;n}$ by $\partial_\lambda V_\lambda$, $|X_n^{(2)}\rangle$, and $m_{2;n}$, respectively. Thus we can write the FF sum rule as

$$f(t)=\{1-t^2/(m_K^2-m_\pi^2)^2+[g^{(2)}(t)]^2-[g^{(0)}(t)]^2\}^{1/2},\tag{5}$$

where

$$[g^{(0)}(t)]^2\equiv\sum_n\frac{\beta_n^{(0)}(t)}{\beta^{(K)}(t)}[g_n^{(0)}(t)]^2,\quad [g_n^{(2)}(t)]^2\equiv\sum_n\frac{\beta_n^{(2)}(t)}{\beta^{(K)}(t)}[g_n^{(2)}(t)]^2,\tag{6}$$

and where it remains to specify the "leakage" functions $g^{(0)}(t)$ and $g^{(0)}(t)$.

We note first that $f_+(m_e^2)$ is given experimentally by[5]

$$0.95\leq f_+(m_e^2)\leq 1.05,\tag{7}$$

so that, to sufficient accuracy, we can take

$$[g^{(2)}(0)]^2-[g^{(0)}(0)]^2=[f(0)]^2-1=[f_+(0)]^2-1=0.\tag{8}$$

Further, the Callan-Treiman (CT) relation[6]

$$f(t=m_K^2;\,p_K^2=-m_K^2;\,p_\pi^2=0)=a_K/a_\pi=1.28,\tag{9}$$

with $a_K, a_\pi\equiv K_{l2}, \pi_{l2}$ decay amplitudes, respectively, when extrapolated onto the pion mass shell according to

$$f(t=[m_K-(p_\pi^2)^{1/2}]^2;\,p_K^2=-m_K^2;\,p_\pi^2)=1+(a_K/a_\pi-1)[1-(-p_\pi^2)^{1/2}/m_K]\tag{10}$$

yields

$$f(t=[m_K-m_\pi]^2;\,p_K^2=-m_K^2;\,p_\pi^2=-m_\pi^2)\equiv f([m_K-m_\pi]^2)=1+(a_K/a_\pi-1)(1-m_\pi/m_K)=1.20,\tag{11}$$

our extrapolation being the simplest which can be arranged to satisfy the SU(3) condition

$$\lim_{\substack{(-p_\pi^2)^{1/2}\to m\\ m_K\to m}} f(t=[m_K-(-p_\pi^2)^{1/2}]^2;\,p_K^2=-m_K^2;\,p_\pi^2)=f(0;-m^2;-m^2)=1.\tag{12}$$

Thus, using Eqs. (11) and (5), we have

$$\{g^{(2)}([m_K-m_\pi]^2)\}-\{g^{(0)}([m_K-m_\pi]^2)\}^2=\left[1+\left(\frac{a_K}{a_\pi}-1\right)\left(1-\frac{m_\pi}{m_K}\right)\right]^2-1+\left(\frac{1-m_\pi/m_K}{1+m_\pi/m_K}\right)^2\equiv a=0.73,\tag{13}$$

consistent with the SU(3) expectation that $[g^{(2)}(t)]^2-[g^{(0)}(t)]^2$ is $O((1-m_\pi/m_K)^2)$.

With $[g^{(2)}(t)]^2-[g^{(0)}(t)]^2$ varying from 0 to 0.73 as t varies from 0 to $(m_K-m_\pi)^2$, we parametrize $g^{(0)}(t)$ and $g^{(2)}(t)$ as linear functions of t, viz.,

$$g^{(0),(2)}(t)=\frac{a^{1/2}(1+t/\langle m_{\mp1;n}\rangle^2)}{\{[1+(m_K-m_\pi)^2/\langle m_{+1;n}\rangle^2]^2-[1+(m_K-m_\pi)^2/\langle m_{-1;n}\rangle^2]^2\}^{1/2}},\tag{14}$$

which satisfy the "constraint" Eqs. (8) and (13) identically for any values of the "average" masses $\langle m_{\mp 1;n}\rangle$. The linearity of the $g^{(0),(2)}(t)$ with t in Eq. (14) corresponds to the assumption that the important $m_{0,2;n}$ are not too different from m_K so that $t_n^{(0),(2)}(t)\approx t$, and to the further assumption that the $Q=S=\mp 1$ intermediate states $|\bar{X}_n^{(0)}\pi^-\rangle$ and $|X_n^{(2)}\pi^-\rangle$ which contribute predominantly to the $g^{(0),(2)}(t)$ have masses $m_{\mp 1;n}$ large compared with m_K. This last assumption also justifies the neglect henceforth of terms $O([(m_K-m_\pi)/\langle m_{\mp 1;n}\rangle]^4)$.[7]

Combining Eqs. (14), (5), and (3) we obtain and expression for $f_-(t)$ which, as expected from SU(3), is $O(1-m_\pi/m_K)$, and which constitutes our basic result, viz.,

$$f(t)=\frac{(m_K^2-m_\pi^2)}{t}\left\{\left[1-\frac{t^2}{(m_K^2-m_\pi^2)^2}+\frac{at}{(m_K-m_\pi)^2}\right]^{1/2}-f_+(t)\right\}. \tag{15}$$

Equation (15) shows that, within the linear, large $m_{\mp 1;n}$ approximation for $g^{(0),(2)}(t)$, the form factor $f_-(t)$ contains no undetermined parameters. For convenience, we record the particular values

$$f_-([m_K-m_\pi]^2)=\frac{1+m_\pi/m_K}{1-m_\pi/m_K}\left\{\left[1+\left(\frac{a_K}{a_\pi}-1\right)\left(1-\frac{m_\pi}{m_K}\right)\right]-\left[1+\lambda_+\left(1-\frac{m_\pi}{m_K}\right)^2+\lambda_+'\left(1-\frac{m_\pi}{m_K}\right)^4+\cdots\right]\right\}=0.12, \tag{16}$$

$$f_-(0)=\xi f_+(0)=\xi=\left(1-\frac{m_\pi^2}{m_K^2}\right)\left[\frac{a}{2}\left(1-\frac{m_\pi}{m_K}\right)^{-2}-\lambda_+\right]=0.42, \tag{17}$$

$$\lambda_-=\frac{(1-m_\pi^2/m_K^2)}{\xi}\left[-\frac{1}{2}\left(1-\frac{m_\pi^2}{m_K^2}\right)^{-2}-\frac{a^2}{8}\left(1-\frac{m_\pi}{m_K}\right)^{-4}-\lambda_+'\right]=-1.8, \tag{18}$$

$$\lambda_-'=\frac{(1-m_\pi^2/m_K^2)}{\xi}\left[\frac{a}{4}\left(1-\frac{m_\pi^2}{m_K^2}\right)^{-2}\left(1-\frac{m_\pi}{m_K}\right)^{-2}+\frac{a^3}{16}\left(1-\frac{m_\pi}{m_K}\right)^{-6}-\lambda_+''\right]=1.2, \tag{19}$$

and

$$\langle\lambda_-\rangle\equiv\frac{f_-([m_K-m_\pi]^2)-f_-(0)}{(1-m_\pi/m_K)^2}=-0.57, \tag{20}$$

where we have used the above quoted numerical magnitudes of a and $f_+(0),\lambda_+,\lambda_+',\cdots$. We note that these values of $\xi,\lambda_-,\lambda_-',\cdots,\lambda_+,\lambda_+',\cdots$ reproduce the observed $K_{\mu 3}/K_{e3}$ branching ratio but predict a $K_{\mu 3}$ transverse polarization roughly twice that observed.

In conclusion, we remark that neglect of the "leakage" matrix elements corresponds to setting $g^{(0),(2)}(t)=0$, and so to setting $a=0$ [Eqs. (3)-(6) and (13)] which reduces the expressions for $f_-(t)$, ξ, and λ_- in Eqs. (15), (17), and (18) to those already given by d'Espagnat and Gaillard.[3] It is clear, however, that neglect of the "leakage" terms is quite inconsistent with the CT relation of Eqs. (9)-(11), since setting $g^{(0),(2)}(t)=0$ requires that $f([m_K-m_\pi]^2)=0.82$ [Eq. (5)] rather than 1.20 [Eq. (11)]. We also note that combination of the CT relation in Eq. (11) with Eqs. (3) and (8) yields

$$\xi=\frac{1+m_\pi/m_K}{1-m_\pi/m_K}\,\frac{\{[1+(a_K/a_\pi-1)(1-m_\pi/m_K)]-[1+\lambda_+(1-m_\pi/m_K)^2+\lambda_+'(1-m_\pi/m_K)^4+\cdots]\}}{[1+\lambda_-(1-m_\pi/m_K)^2+\lambda_-'(1-m_\pi/m_K)^4+\cdots]}. \tag{21}$$

Equation (21) shows, in view of the uncertainties both in the measured values of $a_K/a_\pi,\lambda_+,\lambda_+',\cdots$, and in the predicted values of $\lambda_-,\lambda_-',\cdots$, that it is very difficult to extract a meaningful value of ξ directly from the CT relation.

*Work supported in part by the U. S. Atomic Energy Commission and by the U. S. National Science Foundation.

[1]See the summary in L. B. Auerbach, A. K. Mann, W. K. McFarlane, and F. J. Sciulli, Phys. Rev. Letters 19, 464 (1967).

[2]See, for example, P. Dennery and H. Primakoff, Phys. Rev. 131, 1334 (1963).

[3]B. d'Espagnat and M. K. Gaillard, Phys. Letters 25B, 346 (1967).

[4]S. Fubini and G. Furlan, Physics 1, 229 (1965). See also G. Furlan, F. G. Lannoy, C. Rossetti, and G. Segrè, Nuovo Cimento 38, 1747 (1965).

[5]See the discussion in L. B. Auerbach, J. M. Dobbs, A. K. Mann, W. K. McFarlane, D. H. White, R. Cester, P. T. Eschstruth, G. K. O'Neill, and D. Yount, Phys. Rev. 155, 1505 (1967).

[6]C. G. Callan and S. B. Treiman, Phys. Rev. Letters 16, 153 (1966).

[7]Relaxation of the assumption that $m_{0,2;n} \approx m_K$ and hence that $t_n^{(0),(2)}(t) \approx t$ introduces, in general, the unknown parameters $m_{0,2;n}$ into $g_n^{(0),(2)}(t)$ and $\beta_n^{(0),(2)}(t)$. However, since it is necessary only to cover the range from 0 to 0.73 for $[g^{(2)}(t)]^2-[g^{(0)}(t)]^2$, a linear parametrization for $g^{(0),(2)}(t)$ vs t with a small slope should still be reasonable, even though this slope would no longer be simply $\langle m_{\mp 1;n}\rangle^{-2}$ but would also depend on $\langle m_{0,2;n}\rangle/m_K$. Our result for $f_-(t)$ would nevertheless remain unchanged because, as seen from Eqs. (14) and (15), $f_-(t)$ does not depend on the slope.

CROSS-SECTION SUM RULES FOR NEUTRINO (ANTINEUTRINO) SCATTERING BY NUCLEI FROM CURRENT COMMUTATION RELATIONS*

C. W. Kim
Department of Physics, The Johns Hopkins University, Baltimore, Maryland

and

Michael Ram
Department of Physics, The State University of New York, Buffalo, New York

(Received 4 October 1967)

Total differential cross-section sum rules for forward neutrino (antineutrino) scattering by nuclei are derived using the weak charge commutation relations conjectured by Gell-Mann on the basis of the quark model, and the closure approximation. These sum rules can now be tested with neutrino (antineutrino) beams available at Brookhaven National Laboratory and CERN.

In this Letter, we wish to apply the method recently developed[1] in the calculation of the total muon capture rate in He^3 to derive sum rules for the total differential cross section for forward scattering of neutrinos (antineutrinos) by nuclei. In applying the method we make use of the modifications suggested by Primakoff.[2] The sum rules we derive are the following:

$$\frac{d\sigma^{(\nu)}(A,I,-I_3;\theta=0)}{d\cos\theta}-\frac{d\sigma^{(\nu)}(A,I,I_3;\theta=0)}{d\cos\theta}\cong 8I_3\frac{G^2\cos\theta_C{}^2\nu^2}{\pi}\quad (I_3>0); \tag{1a}$$

$$\frac{d\sigma^{(\bar\nu)}(A,I,I_3;\theta=0)}{d\cos\theta}-\frac{d\sigma^{(\bar\nu)}(A,I,-I_3;\theta=0)}{d\cos\theta}\cong 8I_3\frac{G^2\cos\theta_C{}^2\nu^2}{\pi}\quad (I_3>0); \tag{1b}$$

where $G=(1.02/m_p{}^2)\times10^{-5}$ and $\cos\theta_C=0.98$. In the above, $d\sigma^{(\nu \text{ or } \bar\nu)}(A,I,I_3;\theta)/d\cos\theta$ is the total $\Delta S=0$ differential cross section for the reaction $(\nu_l \text{ or } \bar\nu_l)+N_a\to(l \text{ or } \bar l)+N_b$. In our notation, $N_a=(A,I,I_3)$ is the I_3 member (I_3 is the third component of isospin) of a nuclear isospin multiplet of isospin I and mass number A, and N_b is any allowed final state of hadrons with zero strangeness. The angle θ is the angle of the final lepton $l=e$ or μ relative to the original neutrino (antineutrino) direction, ν is energy of the incident neutrino (antineutrino), and m_p is the proton mass.

We now briefly sketch the essential parts of the derivation, leaving details for a future paper. Using the current-current-type Hamiltonian for weak interactions, one can show that

$$\frac{d\sigma^{(\nu)}(N_a;\theta)}{d\cos\theta}=\sum_b\frac{d\sigma^{(\nu)}(N_a\to N_b;\theta)}{d\cos\theta}=\frac{(G\cos\theta_C)^2}{2\pi}\frac{1}{(2J_a+1)}\sum_b l^2\frac{(E-l)}{(E-V\cos\theta)}\mathcal{L}_{\alpha\beta}{}^{(\nu)}\mathfrak{M}_{\alpha\beta}{}^{(\nu)}(N_a\to N_b). \tag{2}$$

We are assuming that the lepton mass can be neglected so that its four-momentum is simply $p_l\cong(\vec{l},il)$.