FINITE-ENERGY SUM RULES AND BOOTSTRAPS*

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By saturating finite-energy sum rules even at high energies by narrow resonant states, the couplings and the corresponding decay rates of these high resonant states to lower mass particles can be determined in a bootstraplike manner.

Finite-energy sum rules (FESR) were first discussed by Igi[1] and have recently been revived by numerous authors.[2,3] In particular, Dolen, Horn, and Schmid[3] have suggested that FESR might be used for bootstraplike calculations. In this paper we treat a specific case and show that from FESR, given a mesonic Regge trajectory $\alpha(t)$, we can indeed compute its residuum function $\beta(t)$ for large $t(\gtrsim 4\ \mathrm{BeV}^2)$. We consider the amplitude

$$A(s,t)=A_{\pi^-\pi^+\to\pi^-\pi^+}(s,t)-A_{\pi^+\pi^+\to\pi^+\pi^+}(s,t). \quad (1)$$

$A(s,t)$ obeys the FESR

$$\int_0^N \mathrm{Im}A(\nu,t)d\nu=\beta(t)[1+\alpha(t)]^{-1}(N)^{\alpha(t)+1}, \quad (2)$$

where $\nu=\frac{1}{2}(s-u)$ [rather than the usual $\nu=\frac{1}{4}(s-u)$] and $\alpha(t)$ and $\beta(t)$ are the ρ-Regge trajectory and its residuum. We now saturate the left-hand side of Eq. (2) at $t=0$ with single-particle intermediate states. As intermediate states we choose particles of[4]

$$Y=0,\quad I=1,\quad J^P=1^-,3^-,5^-,\cdots,$$
$$Y=0,\quad I=1,\quad J^P=2^+,4^+,\cdots, \quad (3)$$

the particle of spin J having mass m_J. As an alternative to (3) we will also consider what may be called an "oscillator-like" spectrum:

$$Y=0,\quad I=1,\quad J^P=(1^-),\begin{pmatrix}3^-\\1^-\end{pmatrix},\begin{pmatrix}5^-\\3^-\\1^-\end{pmatrix},\cdots,$$
$$Y=0,\quad I=0,\quad J^P=\begin{pmatrix}2^+\\0^+\end{pmatrix},\begin{pmatrix}4^+\\2^+\\0^+\end{pmatrix},\cdots, \quad (4)$$

with all particles in a parenthesis degenerate in mass of mass m_J and correspondingly described by a reducible tensor of rank J. As a rule, we shall present our discussion on the case (3) and then indicate the changes in the result effected by considering (4) instead. In order to have a well-defined problem, we must still make two assumptions. First, we assume that for large J

$$m_J^{\,2}\xrightarrow[J\gg1]{}CJ^\tau \quad (C>0,\ \tau>0) \quad (5)$$

with the same C and τ for J both even and odd [this applies to both cases (3) and (4)].[5] Second, we make a narrow-resonance approximation:

$$\Gamma_J\ll m_J-m_{J-1}\approx dm_J/d_J, \quad (6)$$

where Γ_J is the width of the particle of mass m_J in case (3), and the maximum of the widths of particles of mass m_J in case (4). We define the couplings $J\to\pi^-\pi^+$ in the form

$$g_J P_{\mu_1}\cdots P_{\mu_J}T^{\mu_1\cdots\mu_J}, \quad (7)$$

where T describes the particle [or particles, in case (4)] of mass m_J and $P=\rho_{\pi^+}-\rho_{\pi^-}$. Then, saturating (2) with the intermediate states (3) and choosing $N=CJ_0^{\ \tau}$ with J_0 sufficiently large for (2) to make sense, we find

$$\sum_{J=0}^{J_0} g_J^{\,2}\frac{2J+1}{2^{J+1}}(m_J^{\,2}-4m_\pi^{\,2})^J\pi^{\frac{1}{2}}\frac{\Gamma(J+1)}{\Gamma(J+\frac{3}{2})} =\frac{\beta(0)m_{J_0}^{\,2\alpha(0)+2}}{\alpha(0)+1}. \quad (8)$$

Subtracting from this the similar equation for $N=C(J_0+1)^\tau$, we find

$$g_{J_0}^{\,2}=\frac{\Gamma(J_0+\frac{3}{2})}{\pi^{1/2}\Gamma(J_0+1)}\beta(0)\tau\frac{2^{J_0}(m_{J_0}^{\,2})^{\alpha+1-J_0}}{J_0^{\,2}}, \quad (9)$$

whence the partial width for the decay of the particle of spin J into $\pi^-\pi^+$ is computed to be

$$\Gamma_{J\to\pi^+\pi^-}\approx\frac{\beta(0)\tau}{32\pi}\frac{m_J^{\,2\alpha+1}}{J^2}. \quad (10)$$

Noting that $\Gamma_{J\to\pi^+\pi^-} \leq \Gamma_J$ (the total widths) and comparing with (5) and (6), we find the self-consistency requirement

$$\tau \leq 2. \tag{11}$$

Experimentally, it appears that

$$\tau \approx 1, \tag{12}$$

so that (11) is indeed satisfied. The combination $\beta(0)s^{\alpha(0)}$ can be expressed in terms of total cross sections:

$$\beta(0)N^{\alpha(0)-1} \approx \beta(0)m_{J_0}^{2\alpha(0)-2}$$

$$= (\sigma_{\pi^-\pi^+} - \sigma_{\pi^+\pi^+})_{s=m_{J_0}^2}. \tag{13}$$

From Reggeized ρ universality[6] or the quark model,

$$\sigma_{\pi^-\pi^+} - \sigma_{\pi^+\pi^+} = 2(\sigma_{\pi^-p} - \sigma_{\pi^+p}). \tag{14}$$

Equations (10, (12), and (14) then lead to

$$\Gamma_{J\to\pi^-\pi^+} = (\tau/16\pi)(\sigma_{\pi^-p} - \sigma_{\pi^+p})_{s=m_J^2} m_J^3/J^2. \tag{15}$$

We record here the corresponding equation for the oscillator spectrum (4):

$$\Gamma_{J\to\pi^-\pi^+}^{\mathrm{osc}} = (\tau/16\pi)(\sigma_{\pi^-p} - \sigma_{\pi^+p})_{s=m_J^2} \times (m_J^3/J^2)[(\pi J)^{\frac{1}{2}} 2^{-J}]. \tag{16}$$

Just as an orientation, let us insert numbers into Eqs. (15) and (16). Let us assume the $T(2195)$ meson to have $J^P = 5^-$ and thus correspond to the second Regge recurrence of the ρ^- meson [note that $m_5 = 2.2$ BeV is sufficiently large for (2) to make sense] and that $\tau \approx 1$. Using the experimental values of $\sigma_{\pi^-p} - \sigma_{\pi^+p}$, we then find

$$\Gamma_{5\to\pi^+\pi^-} = 63 \text{ MeV},$$

$$\Gamma_{5\to\pi^+\pi^-}^{\mathrm{osc}} = 7.8 \text{ MeV}, \tag{17}$$

and experimentally the latter result seems to be favored. (Reasonable results can actually be obtained even for $J < 5$, possibly even for the ρ.)

We now wish to give an interpretation of our result. The inegrand of the left-hand side of Eq. (2) is rapidly varying because of resonances; whereas that of the right-hand side is smooth. Equation (2) just states that, in the average, these two functions behave alike (the similarity of this requirement to the standard method in the continuum theory of nuclear reactions is worth noting). It is now clear why $\Gamma_{J\to\pi^-\pi^+}^{\mathrm{osc}} < \Gamma_{J\to\pi^-\pi^+}$. In the oscillator case there are more states competing to average the same right-hand side than in the case of the spectrum (3). Moreover, since we took $t = 0$, all these contributions will be positive.

The method can be easily extended to $t \neq 0$. This is most conveniently done by taking the derivatives of Eq. (2) at $t = 0$. However, at each step of taking one higher derivative, one introduces new parameters, the derivatives of α and β at $t = 0$. Of course, after taking infinitely many derivatives, information on β is obtained, the consistency of which with our result (9) [which essentially determines $\beta(t)$ for large t] has to be tested. For obvious reasons we are not going to make this test here.

We still wish to make a short comment on the amplitude

$$B(s,t) = A_{\pi^-\pi^+\to\pi^-\pi^+}(s,t) + A_{\pi^+\pi^+\to\pi^+\pi^+}(s,t). \tag{18}$$

Should one undertake to saturate an FESR for B with the same sets of intermediate states (3) or (4), one would have on the right-hand side as the leading term the Pomeranchuk contribution. It is easy to check [the reason being that $A_{\pi^+\pi^+\to\pi^+\pi^+}$ does not saturate with single-particle intermediate states (as they ought to have $I = 2$)] that one would obtain a contradiction with Eq. (8) unless $\alpha_\rho(0) = \alpha_{\mathrm{P}}(0)$ which is not the case in reality. This suggests that FESR's which contain a Pomeranchuk contribution on their right-hand sides must have contributions from continuum states on their left-hand sides and cannot be saturated with single-particle states.[8] However, for amplitudes like A that do not contain Pomeranchuk contributions, a saturation of FESR with single-particle intermediate states as we have shown leads to reasonable bootstrap-type results.

Note added in proof.—Assuming the continuum contribution to the FESR for the amplitude B to be confined to the Pomeranchukon, one could subtract the Pomeranchuk contribution on the right-hand side and the continuum contribution on the left-hand side of the FESR. The remainder on the right-hand side would then be dominated by the f trajectory. Saturating the remainder on the left-hand side with the same single-particle intermediate states (3) or (4), we can find the conditions for compatibility with (8). These turn out to have the same form as the conditions that the f trajectory belong to a nonet of trajectories exchange-degenerate with the vector-nonet trajectories in an SU(3)-symmetric world.

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[1]K. Igi, Phys. Rev. Letters 9, 76 (1962).

[2]K. Igi and S. Matsuda, Phys. Rev. Letters 18, 625 (1967); L. D. Soloviev, A. A. Logunov, and A. N. Tavkhelidze, Phys. Letters 24B, 181 (1967).

[3]R. Dolen, D. Horn, and C. Schmid, Phys. Rev. Letters 19, 402 (1967).

[4]If we still include 0^+ mesons in the beginning of the $I=0$ sequence in (3) and (4), this would not at all affect our results. The inclusion of a further $I=0$ sequence starting with 0^+ would affect the result and make the widths of Eq. (10) smaller. The method, however, remains unchanged.

[5]This part of the assumption could be easily relaxed.

[6]P. G. O. Freund, Phys. Rev. Letters 15, 925 (1965).

[7]A. H. Rosenfeld et al., University of California Radiation Laboratory Report No. UCRL-8030, 1967 (unpublished).

[8]This conclusion would have to be modified if higher isospin ($I \geq 2$) and hypercharge ($Y \geq 2$) mesons or baryons should be discovered.