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LOCALIZED (BALLOONING) MODES IN MULTIPOLE CONFIGURATIONS

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(Received 9 November 1967)

We report here an analysis of unstable modes which take place in multipole configurations and are localized along the magnetic field lines. These types of configuration are characterized by having magnetic field curvature and strength varying periodically in such a way that they are, on the average, stable against ordinary interchange modes. But there remains the possibility that new types of modes, localized in regions where the magnetic field profile is unfavorable to forms of instabilities which are not described by the ordinary fluid approximation, may arise.[1] Here we consider modes driven by a transverse density gradient and the longitudinal electron temperature, finding that they are likely to arise around the points of maximum of the (unfavorable) magnetic curvature or of minimum magnetic field (i.e, maximum ion polarization and finite Larmor-radius drift). We restrict ourselves to treating configurations with closed lines of force, although the localized feature of the modes we describe makes possible their existence in more complex configurations.[2] In particular, we use a guiding-center approximation[3] which, for the present case, gives a correct treatment of a realistic configuration and at the same time provides a better understanding of the physical ingredients for the instabilities we present.

We consider a low-pressure situation ($\beta \ll 1$) and look for electrostatic modes the growth rate of which depends on electron-wave resonance effects. So with φ as the electrostatic potential ($E = -\nabla\varphi$), and in the linearized approximation, we look for normal-mode solutions of the form $\varphi = \varphi_1(\chi, \psi) \exp(i\omega t + im\theta)$. In fact we treat a toroidal configuration and adopt the coordinates θ, the angle around the torus; χ, the magnetic potential such that $\vec{B} = \nabla\chi$; and ψ, such that $2\pi\psi$ is the flux of the (poloidal) field contained within a magnetic surface.[2] We also assume a Maxwellian isothermal equilibrium represented to lowest order by the distribution function $f_j = \exp(-E/T_j)p(\psi)$. In particular we choose to consider modes whose effective phase velocity along the lines of force is larger than the ion thermal velocity and smaller than the electron thermal velocity, so that $v_{\mathrm{th}i} > \omega B\varphi_1/|\vec{B}\cdot\nabla\varphi_1| < v_{\mathrm{th}e}$. This will imply that $\omega < \bar{\omega}_{\mathrm{b}e}$, $\bar{\omega}_{\mathrm{b}e}$ being the average bouncing frequency for trapped electrons. Under these conditions we know[1] that the imaginary part of the frequency is small in comparison with the real part and that the stability properties and the topology of the relevant modes are determined, to lowest order, by the real part of the frequency.

Because of all this we can adopt for both the electrons and the ions proper fluid approximations, the results of which we have checked also by direct integration of the Vlasov equation along particle orbits. Considering the ions at first, we have

$$i\omega n_{1G} + \nabla\cdot(n_{1G}\vec{u}) + \nabla\cdot(n_G\vec{u}_{1\perp}) + \nabla\cdot[n_G u_{1\parallel}(\vec{B}/B)] = 0 \quad (1)$$

for mass conservation in the linear approximation. Here the subscript 1 indicates perturbed quantities, n_G the guiding-center density, and $\vec{u}$ the total drift. Then we recall that $\vec{u}=u_{gi}\vec{e}_\theta$, the magnetic curvature drift,

$$\vec{u}_{1\perp}=(1+\tfrac{1}{4}a_i^2\nabla_\perp^2)\vec{E}\times\vec{B}c/B^2+(c/\Omega_iB)d\vec{E}/dt,$$

where $\frac{1}{2}a_i^2=T_i/M_i\Omega_i^2$, Ω_i is the ion gyrofrequency, and $u_{1\parallel}$ is given to lowest order by the equation of motion

$$i\omega nM_iu_{1\parallel}=-\nabla_\parallel(n_1T_i+en\varphi_1), \tag{2}$$

where $\nabla_\parallel=\nabla\chi(\partial/\partial\chi)\equiv(\partial/\partial l)$. In deriving Eq. (2) we have anticipated that the modes we consider have frequency $\omega\approx m(cT_e/ep)(dp/d\psi)=-m\omega_{\mathrm{d}e}$, $\omega_{\mathrm{d}e}$ being the electron diamagnetic frequency, and assumed that $(1/B)(dB/d\psi)\ll(1/p)(dp/d\psi)$, so that $\omega_{\mathrm{d}e}\gg\omega_{gi}\equiv(2cT_i/eB)dB/d\psi$, the curvature drift frequency. Finally we limit attention to modes localized around $\psi=\psi_0$ and such that $|\nabla_\psi\varphi_1|\ll im\varphi_1/R$. Consequently, we treat all quantities as constant in ψ to lowest order and $\nabla_\perp^2=-m^2/R^2$, R being the torus major radius. In addition, we have to lowest order $n_G=n$, the ion density, and $n_{li}=[1-(a_i^2/4)m^2/R^2]n_{lG}$. We can then obtain from Eq. (1)

$$n_{li}=-\frac{en}{T_i}\left\{1-\left(1+m\frac{\omega_{\mathrm{d}i}}{\omega}\right)\left[1+\frac{m}{\omega}\omega_{gi}(\chi)-\frac{T_im^2}{M_iR^2}\frac{1}{\Omega_i^2}(\chi)-\frac{T_i}{M_i\omega^2}B^2\frac{d^2}{d\chi^2}\right]\right\}\varphi_1(\chi). \tag{3}$$

Under the assumption that we have specified before, we can simply write for the electrons

$$\nabla_\perp(n_{1e}T_e-en\varphi_1)=0,$$

so that

$$n_{1e}=(en/T_e)\varphi_1. \tag{4}$$

This result is correct to lowest order if φ_1 is odd in l or if φ_1 is sufficiently localized so that the perturbation can be thought of as having an effective $k_\parallel$ such that $\omega/k_\parallel<v_{\mathrm{th}e}$, the electron thermal velocity. We can see this more rigorously recalling that, since $\omega<\bar{\omega}_{\mathrm{b}e}$, the correction term[2] to Eq. (4) is such that

$$n_{1e}=[\varphi_1-(1+m\omega_{\mathrm{d}e}/\omega)\langle\varphi_1\rangle]en/T_e,$$

where

$$\langle\varphi_1\rangle=n^{-1}\int f_ed^3v\oint\varphi_1\frac{dl}{v_\parallel}\left(\oint\frac{dl}{v_\parallel}\right)^{-1},$$

the latter term indicates average over the electron orbit, and $dl=d\chi/B$.

The dispersion relation is obtained from the condition $\tilde{n}_i=\tilde{n}_e$ and is

$$\left\{\frac{T_i}{M_i\omega^2}B^2\frac{d^2}{d\chi^2}+b(\chi)-\frac{m}{\omega}\omega_{gi}(\chi)-\frac{\omega T_i/T_e-m\omega_{\mathrm{d}i}}{\omega+m\omega_{\mathrm{d}i}}\right\}\varphi_1(\chi)=0, \tag{5}$$

where $b(\chi)\equiv(m/R\Omega_i)^2T_i/M_i\ll1$. Now the condition that all terms inside brackets in Eq. (5) be of the same order leads to having $\omega=m\omega_{\mathrm{d}i}\times(T_e/T_i)+\delta\omega$, where $\delta\omega/\omega\ll1$. The same equation having periodic coefficients can be solved by standard methods. The most interesting solutions, however, are found to be localized in the region where $b(\chi)+T_i\omega_g(\chi)/\omega_{\mathrm{d}i}T_e$ is maximum. This region, depending on the values of m, can be centered around either the point of minimum B or the point of maximum unfavorable curvature. There Eq. (5) reduces to the form

$$\left\{\frac{T_i}{M_im^2\omega_{\mathrm{d}e}^2}\frac{d^2}{dl^2}-\delta\delta\bar{\omega}-\frac{l^2}{2\mathfrak{L}^2}\left(b_0+\frac{T_i}{T_e}\frac{\omega_{gi0}}{\omega_{\mathrm{d}i}}\right)\right\}\varphi_1(l)=0, \tag{6}$$

where $\delta\delta\bar{\omega}=\delta\omega/(m\omega_{\mathrm{d}e})(1+T_e/T_i)-b(l_0)-T_i\omega_{gi}(l_0)/T_e\omega_{\mathrm{d}i}$, and $[b_0+(T_i/T_e)(\omega_{gi0}/\omega_{\mathrm{d}i})]/\mathfrak{L}^2\equiv(d^2/dl^2)[b+(T_i/T_e)(\omega_{gi}/\omega_{\mathrm{d}i})]_{l=l_0}$. The eigensolutions of Eq. (6) are $\varphi_{1\mathrm{n}}=H_n(l/\Delta)\exp(-l^2/2\Delta^2)$, H_n being Hermite polynomials and Δ representing the localization length for these modes. So we obtain

$$\Delta\approx(\mathfrak{L}r)^{1/2}\left(b_0\frac{T_e}{T_i}+\frac{d\ln B}{d\ln p}\right)^{-1/4},$$

where $r^{-1}=-(B/p)dp/d\psi$. Now, for the validity of the present result, we require $\Delta^2<\mathfrak{L}^2$ so that we can give as a stability condition for this localized mode

$$\mathfrak{L}<\frac{r}{b_0}\frac{T_i}{T_e}\left(1+\frac{T_i}{b_0T_e}\frac{d\ln B}{d\ln p}\right)^{-1/2}. \tag{7}$$

If we interpret $\mathcal{L}$ as a measure of the "connection length," i.e., of the distance between good and bad curvature, then we see that unless T_i/T_e is fairly large, the stability criterion Eq. (7) can be quite severe to satisfy.

Then if we consider modes that are not localized but extend over the entire length L_t of the lines of force, we can assume $k_\parallel \approx \pi/L_t$ and impose, for stability, that $|\omega| > m\omega_{\mathrm{d}e}$, so that we obtain from Eq. (5) $\delta\omega = b - k_\parallel^2 T_i/(M_i m^2 \omega_{\mathrm{d}e}^2) < 0$. This corresponds to making the effects of longitudinal ion inertia prevail over the inertia across the field,[1] and we obtain the criterion

$$L_t < \frac{r}{b}\frac{T_i\pi}{T_e}\left(1+\frac{T_i}{bT_e}\frac{d\ln B}{d\ln p}\right)^{-1/2},$$

similar to Eq. (7). We note also that if $\omega < \omega_{\mathrm{b}i}$ then ion Landau damping will lead to stabilization of all such electron drift modes. This gives, for stability,

$$L_t \lesssim r(T_i/T_e)(2/b)^{1/2}\pi.$$

Now we can argue that the localized modes discussed earlier can exist, because of these features, also in configurations with open lines of force and magnetic shear. In particular, we expect that shear should not be able to eliminate the modes until they do not "see" it, i.e., until the shearing distance L_S becomes of the order of Δ or smaller.

Recent experiments on the Princeton linear quadrupole LM-1 have led to the observation of a high-frequency oscillation localized in the region of unfavorable curvature.[4] This appears to be due to an instability of drift type similar to the one discussed here. However, the experimental value of b is rather large in comparison with $(T_i/T_e)d\ln B/d\ln p$. In fact, as reported,[4] b ranges between 0 and $\frac{1}{2}$, whereas $(T_i/T_e)d\ln B/d\ln p \approx 0.01$.

Thus, if we take the localization condition as predicted by the asymptotic theory presented here, in which $b \ll 1$, we should expect to see modes localized in the good-curvature section of the lines of force. For this reason an approximate equation, valid for large b, has been investigated numerically[5] with the intention of finding solutions localized in the unfavorable-curvature section. However, the validity of the approximation used to drive the large-b solution needs further analysis before presenting a definite result.

Finally, we may also argue that, because of the small growth rate of the modes under consideration, we expect a small diffusion coefficient for radial particle transport. Therefore, they could be reasonably tolerated even in the case where, for a given thermonuclear device, the geometric features of the magnetic configuration (connection lengths, curvatures, etc.) or the ion and electron temperatures may not be adequate to achieve complete stability.

For the sake of illustration, we recall that typically in octopole devices $L_t/r \approx 10$, whereas in quadrupole devices $L_t/r \approx 25$. Here we define r as the minimum distance between the stagnation surface (ψ_S) and the critical surface (ψ_C), where

$$\frac{d}{dr}\oint\frac{dl}{B} = 0.$$

This distance is reasonably close to the scale distance for the plasma density gradient considered previously.

In a typical experiment on the mentioned LM-1 quadrupole device, T_i/T_e ranges from 0.1 to 1, and in the octopole experiments of the University of Wisconsin[6] and of General Atomic,[7] T_i/T_e is typically 10. The ratio $d\ln B/d\ln p$, measured at the maximum field at ψ_S, is $\frac{1}{10}$ in LM-1 and about $\frac{1}{5}$ in octopoles.

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