ordinary differential-equation systems.[4]

[1]N. Bogoliubov and Y. A. Mitropolskii, Asymptotic Methods in the Theory of Nonlinear Oscillations (Gordon and Breach Publishers, Inc., New York, 1961), Chap. 3, Sec. 17.

[2]A large number of theoretical and experimental studies have been reported in this field: E. A. Jackson, Phys. Fluids 3, 786 (1960); I. B. Bernstein and R. M. Kulsrud, Phys. Fluids 3, 937 (1960); I. Alexeff and R. V. Neidigh, Phys. Rev. 129, 516 (1963). The oscillations in our experiment were identified with the ion-acoustic waves in the following ways. When the oscillation existed, the frequency of the oscillation as detected at grid G decreased with the increase of the distance L between E_3 and G. The frequency f_a should be, and was observed to be, proportional to L^{-1}. It was also confirmed with the use of a photomultiplier that the ion-acoustic wave was in fact a half-wavelength standing wave between E_2 and G. The phase velocity V_s obtained from the relation between f_a and the wavelength $2(L-2L_s)$, where L_s is a sheath thickness ~0.1 cm, agrees very well with that calculated from probe measurements; for example, $f_a = 60$ kHz, $L-2L_s = 0.8$ cm, then $V_s = 0.96\times10^5$ cm/sec, and the electron temperature $T_e = 2.0$ eV. However, there remains the question whether this ion-acoustic wave is the direct result of a two-stream instability or other mechanism, such as the decay-type instability which was given by V. N. Oraevskii and R. Z. Sagdeev, Zh. Techn. Fiz. 32, 1291 (1962) [translation: Soviet Phys.—Tech. Phys. 7, 955 (1963)], and Y. Ichikawa, Phys. Fluids 9, 1454 (1966), etc.

[3]A similar result has been obtained by E. A. Kornilov et al., Zh. Eksperim. i Teor. Fiz.—Pis'ma Redakt. 3, 354 (1966) [translation: JETP Letters 3, 229 (1966)], who applied microwave signals to an electron beam of about 3 kV for modulation and observed the quenching of the oscillations excited by the beam, but they did not classify the oscillations. The stabilization of a two-stream instability by virtue of the existence of an electric field has been treated theoretically by Yu. M. Aliev and V. P. Silin, Zh. Eksperim. i Teor. Fiz. 48, 901 (1965) [translation: Soviet Phys.—JETP 21, 601 (1965)]. The assumption and the result of this theory are much different from our experiment.

[4]Bogoliubov and Mitropolskii, Ref. 1, Sec. 13.

SUPERFLUIDITY OF LOW-DENSITY FERMION SYSTEMS

D. Fay* and A. Layzer†
Stevens Institute of Technology, Hoboken, New Jersey
(Received 12 December 1967)

In the weak-coupling limit, it has been shown[1] that a system of fermions with purely repulsive forces will be superfluid at zero temperature. This is because the two-particle interaction induced by many-body effects is attractive at the Fermi surface provided that the angular momentum of the pair is sufficiently high. These asymptotic considerations do not permit the specification of the actual momentum of the pairing in the condensed state, much less an accurate estimate of the transition temperature T_c or the magnitude of the energy gap Δ.

We present results here concerning the exact low-density behavior of the pairing interaction U for fermions with arbitrarily strong short-range forces. Only the $T=0$ and $\Delta=0$ limit of U will be considered. It is found that at low densities a pairing interaction can in fact be defined in the BCS manner such that both T_c and Δ are proportional to $\exp[2/\pi m^* \times k_F^{-1}U(k_F,k_F)]$. At low densities, the correction δU induced by many-body effects turns out to be attractive in P states, less strongly attractive in D states, and repulsive in S states. A system of hard spheres at low densities and at $T=0$ should be superfluid in a condensed state with P-state pairing.

Quantitatively we obtain the following results in the limit of low densities. In spite of the strength of the primary interaction v, at low densities δU tends to the expectation value of the induced potential δI on the Fermi surface and on the energy shell. δI is shown in Fig. 1. The blocks T at low densities may be replaced by combinations of ladder diagrams for particle-particle scattering and χ may be replaced by a single particle-hole loop. In the low-density limit with p and k near k_F, $\delta I(p,k)$ turns out to be precisely the same as the lowest-order perturbation value except that, in the final result, $v \to (4\pi/m)\bar{t}_0$, where $\bar{t}_0$ is the zero-energy scattering amplitude ($\bar{t}_0 = R$ for hard spheres). One obtains in this limit

$$\delta I(p,k) = (4\pi/m)^2\bar{t}_0^{\,2}Q(p+k), \tag{1}$$

$$\delta\overline{U}_l = (2/\pi)(k_F\bar{t}_0)^2\overline{Q}_l(0)(-1)^l, \tag{2}$$

where $\overline{U} = mk_F^{-1}U$, Q is one-half the usual

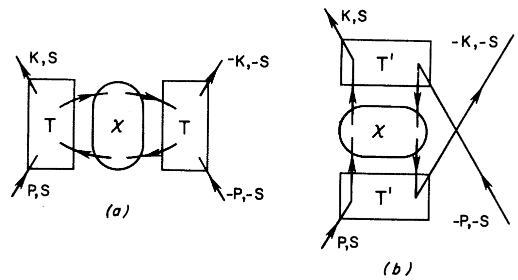

FIG. 1. Diagrammatic representation of induced force $\delta I(p,k)$, containing all corrections to the irreducible interaction function for particles with equal and opposite initial momentum p and spin s. The blocks T represent irreducible particle-hole kernels, while χ is the complete generalized response function. At low densities χ reduces to a single particle-hole loop.

polarization bubble, and $\overline{Q}(0)=(2\pi mk_{\mathrm{F}})^{-1}Q(q,0)$ with $q=p-k$, and is normalized to unity at $q=q_0/q=0$. We note that $\overline{Q}(0)$ can be found in position space by contour integration and is, in terms of spherical Bessel functions,

$$\overline{Q}(r)=\mathrm{const}\,j_1(2k_{\mathrm{F}}r)/(k_{\mathrm{F}}r^2). \tag{3}$$

For $l=0$, 1, and 2, $\overline{Q}_l(-1)^l$ has the following values: $l=0$, $\frac{1}{3}(2\ln 2+1)=+0.80$; $l=1$, $\frac{1}{5}[-2\ln 2+1]=-0.08$; and $l=2$, $(2/105)[-8+11\ln 2]=-0.008$. For asymptotically large l,[1] and therefore presumably for all $l>0$, $Q_l(-1)^l$ is negative and monotonically decreasing in absolute value.

Equation (2) is to be compared with the BCS estimate[2] in terms of a free-space phase shift for relative momentum equal to k_{F},

$$\overline{U}_{\mathrm{BCS}}\cong-\tan\delta_l. \tag{4}$$

This is of order $(k_{\mathrm{F}}\bar{r}_0)^{2l+1}$. The approximation[4] to BCS can be shown to be valid to leading order, for each l. Thus for all $l>0$, δU dominates over the BCS term at sufficiently low densities. Systems with repulsive free-space S-wave scattering at momenta corresponding to these low densities are necessarily in superfluid states with P-state pairing.

Curiously, an inversion of a "classical" model[3] gives the correct sign for the induced force at low densities, attractive for $l>0$. An interaction of the type of Fig. 1(a) between, say, hard spheres A and B is classically a three-body interaction, the third body C being the sphere knocked out of the Fermi sea in the intermediate Feynman loop. The third sphere is effective when it sits between A and B, there serving to deflect B from A, simulating a direct repulsive force. Quantum mechanically, however, internal exchange forces at the T's cancel out the process of Fig. 1(a) entirely at low densities, leaving only Fig. 1(b). In this exchange process B changes identity with C and as a result B is deflected toward A as if by a direct attractive force. For the special case $l=0$, we can say that B comes between A and C, reversing the final deviation.

This argument appears to depend, as it should not, on the primary force between A and B being repulsive. In reality, it depends on a repulsive exchange force between the exchanged particles B and C only, in states of parallel spin. It depends also on the fact that the range k_{F}^{-1} of the exchange force is considerably larger than that of the primary force at low densities.

Consider now the exact situation. For strong interactions, the pairing interaction is determined in BCS by the equation

$$U(k,k')=\langle k|v'|\psi(k')\rangle, \tag{5}$$

where ψ satisfies an integral equation of Schrodinger wave-matrix form, though with modified Green's function $\overline{G}$.[4] In BCS, v' is the bare interaction v. In the exact case one can show the following. At low densities there exist Hermitian and energy-independent U and v' satisfying (5) with Δ and T_C connected exponentially to U as in strong-interaction BCS.[4] $\overline{G}$ is further modified by the expected quasiparticle (q-p) corrections for energy-spectrum, renormalization and lifetime effects.[5] v' has the form

$$v'=v+\mathrm{Re}\,\delta I+M+M^{\dagger}. \tag{6}$$

The expression for M is not unique. At low densities only one has $M(p,k)\cong\mathrm{const}\,|\epsilon(k)|P\int_0^\infty dk_0\times(k_0^2-\epsilon^2)^{-1}\,\mathrm{Im}\,\delta I(p,k)$ with $p_0=\epsilon(p)$. Here ϵ is the q-p energy relative to the Fermi energy. In (6) a three-dimensional momentum representation is understood with matrix elements between states, say p and k. For $\mathrm{Re}\,\delta I$ we must take $p_0=\epsilon(p)$; $k_0=\epsilon(k)$. More specialized results for a density expansion of δU are as follows. In terms of the parameter $x=k_0\bar{r}_{\mathrm{F}}$, q-p corrections enter first at order $x^4\ln x$ for $l=0$ and $x^5\ln x$ for $l>0$.[6] The nonstatic part of $\mathrm{Re}\,\delta I$ and the term M in (6) enter at order x^{2l+3}. An approximate estimate of the x^3 term for $l>0$ has been made. Like the coefficient of the lowest order x^2 term (2), that of the x^3 term is independent of the shape or range of v. Higher-order terms are in general shape dependent.

We very briefly sketch the derivation of the more general results above.[7] From work on strong-interaction BCS, it is sufficient to prove that one can write the exact gap equation in BCS form with v' replacing v and with the appropriate q-p replacement of $|\epsilon|$. Consider the exact integral equation for the gap function $\varphi(p,p_0)$ with kernel $I(p,k)$, to lowest order in the gap $\Delta \equiv \varphi(p_F, 0)$. Take $|p_0| < \Delta$. Then the integrand will have two branch cuts along the real axis of the complex k_0 plane running from $|k_0| = \Delta$ to ∞. Deform the Feynman contour of integration over k_0 to a hairpin loop about the right-hand cut. This yields two contributions. The first is due to the discontinuity, across the cut, of the product $G(k)G(-k)$ of single-particle propagators entering into the gap equation. In a q-p approximation this reduces to a pole term which is of BCS form for $\varphi[k, \epsilon(k)]$. The discontinuity of $I\varphi$ yields a second term which gives rise in lowest order to the M term given below (6), upon making use of the original equation to evaluate the discontinuity.

The derivation indicates that the general results (5) and (6) and the connection of T_C and Δ to U are valid up to and including the order of first appearance of q-p corrections to $\overline{G}$ and of the term M in (6). A formal extension of this proof leads to the stronger conclusion that the general results, except for the specific form of M, are probably true at all densities provided only that one works within the framework of a q-p approximation to the energy-variable discontinuity of the product $G(k)G(-k)$ entering into the exact gap equation.

Physical examples of neutral quantum fermion systems in the low-density region are few. Our results are not directly applicable to dilute He^3 and He^4 mixtures.[8] They are applicable to the neutron gas, a rough model of the hypothesized neutron star. The neutron gas at low densities is expected in BCS theory to be in a condensed state with S-state pairing.[9] The induced effect discussed here reduces the S-state gap and favors a transition to a condensed state with P- or D-state pairing at a density presumably in the neighborhood of nuclear core densities, since the S-state phase shift goes to zero in this region.

In the region of intermediate densities, we underline the importance of strong-interaction effects, effects leading to the deviation of δU from $\delta I(k_F, k_F)$. These appear as early as the order x^4 in the density expansion of δU. In almost all practical cases, one cannot avoid the somewhat cumbersome machinery of solving the integral equation for U. One can, however, over a wide range of densities (at least for $l > 0$) employ a linearized approximation for the solution with the BCS results as lowest order.[10]

It is interesting that the low-density mechanism for the induced force is similar to that in the nearly ferromagnetic region, typified presumably by liquid He^3.[11] In the latter case, as in Fig. 1(b), terms contributing to δI are characterized by having spin unity rather than spin zero in the particle-hole cross channel; though in the nearly ferromagnetic case, smaller noncancelling contributions in this spin channel also arise from Fig. 1(a).[12] This reveals the existence of an intimate connection between a superfluid tendency in higher l states and a ferromagnetic tendency for the system, both at very low and at comparatively high densities.

One of us (A.L) would like to acknowledge the hospitality of J. Weneser and Brookhaven National Laboratory during the summer of 1966.

---

*Stanley Predoctoral Fellow.

†Presently on leave.

[1]J. Luttinger, in 1965 Tokyo Summer Lectures in Theoretical Physics (W. A. Benjamin, Inc., New York, 1966), Pt. I.

[2]V. Emery, Nucl. Phys. 19, 154 (1960).

[3]The original heuristic "classical model" was suggested to us by V. Emery.

[4]V. Emery and A. M. Sessler, Phys. Rev. 119, 43, (1960). A typographical error should be noted in (A4) of that paper, defining a term in $\overline{G}$. This lacks a factor (k_F^2/k^2) in the Fermi surface term. An alternative formalism which we do not employ is due to P. Anderson and P. Morel, Phys. Rev. 123, 1911 (1961).

[5]P. Morel and P. Nozières, Phys. Rev. 126, 1909 (1962).

[6]Moreover, the entire contribution at this order is due to the behavior of the q-p corrections at asymptotically large momenta, $p \gg k_F$.

[7]Details of the proof and of other results cited here will be published elsewhere. Results on the density expansion of δU were proved explicitly for v belonging to the class of three-parameter potentials of the form

$$v(r) = \lim_{\lambda \to \infty} \lambda\delta(r-a) - \lambda'\delta(r-b) \text{ with } b > a,$$

simulating a hard core plus attractive tail. Parameter-independent results are presumed to hold for all well-

behaved potentials of short range.

[8]The mixtures have been analyzed as equivalent to low-density fermion systems with an effective interaction between He^3 atoms: V. J. Emery, Phys. Rev. 148, A138 (1966). However, the level of that approximation does not yet permit a consistent treatment of the induced force of Fig. 1.

[9]K. A. Brueckner, J. L. Gammel, and J. T. Kubis, Phys. Rev. 118, 1095 (1960); P. Sood and S. Moszkowski, Nucl. Phys. 21, 582 (1960).

[10]A. Layzer and D. Fay, to be published.

[11]N. Berk and J. Schrieffer, Phys. Rev. Letters 17, 433 (1966); S. Doniach and S. Engelsberg, Phys. Rev. Letters 17, 750 (1966).

[12]See Ref. 10. The particle-hole spin-channel description is very convenient as soon as one gets away from the low-density limit. In the nearly ferromagnetic limit, the contribution of diagram (a) of Fig. 1 to the static value of δI is $\frac{1}{2}(-1)^l$ that of diagram (b), for individual l states.

LOCALIZED (BALLOONING) MODES IN MULTIPOLE CONFIGURATIONS

B. Coppi and M. N. Rosenbluth
Institute for Advanced Study, Princeton, New Jersey, and Princeton University, Princeton, New Jersey

and

S. Yoshikawa
Plasma Physics Laboratory, Princeton University, Princeton, New Jersey
(Received 9 November 1967)

We report here an analysis of unstable modes which take place in multipole configurations and are localized along the magnetic field lines. These types of configuration are characterized by having magnetic field curvature and strength varying periodically in such a way that they are, on the average, stable against ordinary interchange modes. But there remains the possibility that new types of modes, localized in regions where the magnetic field profile is unfavorable to forms of instabilities which are not described by the ordinary fluid approximation, may arise.[1] Here we consider modes driven by a transverse density gradient and the longitudinal electron temperature, finding that they are likely to arise around the points of maximum of the (unfavorable) magnetic curvature or of minimum magnetic field (i.e, maximum ion polarization and finite Larmor-radius drift). We restrict ourselves to treating configurations with closed lines of force, although the localized feature of the modes we describe makes possible their existence in more complex configurations.[2] In particular, we use a guiding-center approximation[3] which, for the present case, gives a correct treatment of a realistic configuration and at the same time provides a better understanding of the physical ingredients for the instabilities we present.

We consider a low-pressure situation ($\beta \ll 1$) and look for electrostatic modes the growth rate of which depends on electron-wave resonance effects. So with φ as the electrostatic potential ($E = -\nabla\varphi$), and in the linearized approximation, we look for normal-mode solutions of the form $\varphi = \varphi_1(\chi, \psi) \exp(i\omega t + im\theta)$. In fact we treat a toroidal configuration and adopt the coordinates θ, the angle around the torus; χ, the magnetic potential such that $\vec{B} = \nabla\chi$; and ψ, such that $2\pi\psi$ is the flux of the (poloidal) field contained within a magnetic surface.[2] We also assume a Maxwellian isothermal equilibrium represented to lowest order by the distribution function $f_j = \exp(-E/T_j)p(\psi)$. In particular we choose to consider modes whose effective phase velocity along the lines of force is larger than the ion thermal velocity and smaller than the electron thermal velocity, so that $v_{\mathrm{th}i} > \omega B\varphi_1/|\vec{B}\cdot\nabla\varphi_1| < v_{\mathrm{th}e}$. This will imply that $\omega < \bar{\omega}_{\mathrm{b}e}$, $\bar{\omega}_{\mathrm{b}e}$ being the average bouncing frequency for trapped electrons. Under these conditions we know[1] that the imaginary part of the frequency is small in comparison with the real part and that the stability properties and the topology of the relevant modes are determined, to lowest order, by the real part of the frequency.

Because of all this we can adopt for both the electrons and the ions proper fluid approximations, the results of which we have checked also by direct integration of the Vlasov equation along particle orbits. Considering the ions at first, we have

$$i\omega n_{1G} + \nabla\cdot(n_{1G}\vec{u}) + \nabla\cdot(n_G\vec{u}_{1\perp}) + \nabla\cdot[n_G u_{1\parallel}(\vec{B}/B)] = 0 \tag{1}$$