computing assistance.

*Research supported in part by the U. S. Air Force through Grant No. AF-AFOSR-1310-67.

[1]M. E. Fisher, Rept. Progr. Phys. 30, 615, (1967).

[2]T. Matsubara and H. Matsuda, Progr. Theoret. Phys. (Kyoto) 16, 416 (1956).

[3]R. F. Wielinga, J. Lubbers, and W. J. Huiskamp, Physica 37, 375 (1967). We are grateful to Professor Huiskamp for sending us a copy of their results prior to publication.

[4]Ref. 1, Sec. 4.3.

[5]R. Whitlock and P. R. Zilsel, Phys. Rev. 13, 2409 (1963).

[6]C. N. Yang and C. P. Yang, Phys. Rev. Letters 13, 303 (1964), and Phys. Rev. 147, 303 (1966).

[7]T. Obokata, I. Ono, and T. Oguchi, J. Phys. Soc. Japan 23, 516 (1967).

[8]K. Pirnie, unpublished. We are grateful to Dr. Pirnie for informing us of his results prior to publication.

[9]C. Domb, Advan. Phys. 9, 149 (1960).

[10]M. F. Sykes, J. W. Essam, B. R. Heap, and B. J. Hiley, J. Math. Phys. 7, 1557 (1966).

[11]G. A. Baker, H. E. Gilbert, J. Eve, and G. S. Rushbrooke, Brookhaven National Laboratory Report No. BNL-50053 (T-460), 1967 (unpublished).

[12]C. Domb and M. F. Sykes, Proc. Roy. Soc. (London), Ser. A 240, 214 (1957); Ref. 1, Sec. 7.2.

[13]G. A. Baker, Phys. Rev. 122, 1477 (1961), and in Advances in Theoretical Physics, edited by K. A. Brueckner (Academic Press, Inc., New York, 1965), Vol. I, p. 1.

[14]G. A. Baker, H. E. Gilbert, J. Eve, and G. S. Rushbrooke, Phys. Rev. 164, 800 (1967).

[15]W. M. Fairbank and C. F. Kellers, in Critical Phenomena, Proceedings of a Conference, Washington, D. C., 1965, edited by M. S. Green and J. V. Sengers, National Bureau of Standards Miscellaneous Publication No. 273 (U.S. Government Printing Office, Washington, D. C., 1966), p. 71.

# EXPRESSIONS FOR THE TRANSMISSION COEFFICIENTS AND THE MEAN SQUARE DEVIATION OF THE ELASTIC-SCATTERING CROSS SECTION

Nazakat Ullah
Tata Institute of Fundamental Research, Bombay, India
(Received 1 December 1967; revised manuscript received 27 March 1968)

An expression for the transmission coefficient for the general case of an arbitrary number of open channels, valid for all values of the ratio of the average partial width to the average spacing, is obtained from a completely unitary matrix. For the purely elastic-scattering case, the average value and the mean-square deviation of the elastic-scattering cross section are expressed in terms of the potential-scattering phase shift and the ratio of the average width to the average spacing.

In the statistical study of the low-energy nuclear reactions which pass through the formation of a compound nucleus, one tries to express the average quantities like the transmission coefficients[1] and the average cross sections[2] in terms of the averages of the resonance parameters of the low-energy collision matrix. The resonance parameters which enter into the expressions for the average quantities are the partial widths and the spacings of the poles of the collision matrix. The two main difficulties in deriving these results have been (1) keeping track of the unitary condition on the resonance-pole expansion form of the low-energy collision matrix, and (2) validity of the expressions for all values of the ratio of the average partial width to the average spacing. Our aim in this note is to show that exact expressions for the transmission coefficients and the average cross sections can be obtained if we start from the pole-resonance form of the unitary matrix, which has been given recently by us.[3]

We have recently shown[3] that the unitary pole-resonance form of the collision matrix $U$, based on $R$-matrix theory[4] or Feshbach's unified theory of nuclear reactions,[5] can always be written as

$$U(E)=V\left[1-i\sum_{\mu=1}^{N}\frac{G_\mu}{E-z_\mu}\right]V, \tag{1}$$

where $z_\mu=\mathcal{E}_\mu-\frac{1}{2}i\Gamma_\mu$ are the complex poles, and the matrix elements of $G_\mu$ are the complex width amplitudes. The matrix $V$ gives rise to direct reactions and for the case of $m$ channels can be written as

$$V=Od\tilde{O}, \tag{2}$$

where $O$ is an $m\times m$ real orthogonal matrix and $d$ is a diagonal matrix, the elements of which are $d_{cc'}=\exp(-i\varphi_c)\delta_{cc'}$.

We first consider the purely elastic-scattering case ($m=1$); then the complex quantities $G_\mu$ are given by[3]

$$G_\mu = \Gamma_\mu \prod_{\nu\neq\mu}^{N} (z_\mu - z_\nu^*)(z_\mu - z_\nu)^{-1}. \quad (3)$$

Following Feshbach, Kerman, and Lemmer,[6] we use the Lorentz weighting function

$$\rho(E,E_0) = \frac{I}{2\pi}\frac{1}{(E-E_0)^2+\frac{1}{4}I^2}, \quad (4)$$

with

$$I \sim 2\Delta E/\pi,$$

and write the energy average of the collision function given by expression (1) as

$$\langle U(E)\rangle = \int_{\Delta E}\rho(E,E_0)U(E)dE. \quad (5)$$

Using expressions (1), (3), and (4), we get from expression (5)

$$\langle U\rangle = \exp(-2i\varphi)\prod_{\mu=1}^{N}\frac{E_0-\mathcal{E}_\mu+\frac{1}{2}i(I-\Gamma_\mu)}{E_0-\mathcal{E}_\mu+\frac{1}{2}i(I+\Gamma_\mu)}. \quad (6)$$

Expanding the numerator and the denominator in expression (6) in powers of $1/I$, we get

$$\langle U\rangle = \exp(-2i\varphi)\frac{1+\frac{2i}{I}\sum_{\mu=1}^{N}(\mathcal{E}_\mu-E_0+\frac{1}{2}i\Gamma_\mu)+\left(\frac{2i}{I}\right)^2\sum_{\mu<\nu}(\mathcal{E}_\mu-E_0+\frac{1}{2}i\Gamma_\mu)(\mathcal{E}_\nu-E_0+\frac{1}{2}i\Gamma_\nu)+\cdots}{1+\frac{2i}{I}\sum_{\mu=1}^{N}(\mathcal{E}_\mu-E_0-\frac{1}{2}i\Gamma_\mu)+\left(\frac{2i}{I}\right)^2\sum_{\mu<\nu}(\mathcal{E}_\mu-E_0-\frac{1}{2}i\Gamma_\mu)(\mathcal{E}_\nu-E_0-\frac{1}{2}i\Gamma_\nu)+\cdots}. \quad (7)$$

The second, third, $\cdots$, terms in numerator and denominator are rewritten using the relations of the type

$$\left[\sum_{\mu=1}^{N}(\mathcal{E}_\mu-E_0+\frac{1}{2}i\Gamma_\mu)\right]^2 = \sum_{\mu=1}^{N}(\mathcal{E}_\mu-E_0+\frac{1}{2}i\Gamma_\mu)^2 + 2\sum_{\mu<\nu}(\mathcal{E}_\mu-E_0+\frac{1}{2}i\Gamma_\mu)(\mathcal{E}_\nu-E_0+\frac{1}{2}i\Gamma_\nu).$$

Expression (7) now becomes

$$\langle U\rangle = \exp(-2i\varphi)$$

$$\times\frac{1+\frac{2i}{I}\sum_{\mu=1}^{N}(\mathcal{E}_\mu-E_0+\frac{1}{2}i\Gamma_\mu)+\frac{1}{2!}\left(\frac{2i}{I}\right)^2\left[\sum_\mu(\mathcal{E}_\mu-E_0+\frac{1}{2}i\Gamma_\mu)\right]^2-\frac{1}{2!}\left(\frac{2i}{I}\right)^2\sum_\mu(\mathcal{E}_\mu-E_0+\frac{1}{2}i\Gamma_\mu)^2+\cdots}{1+\frac{2i}{I}\sum_{\mu=1}^{N}(\mathcal{E}_\mu-E_0-\frac{1}{2}i\Gamma_\mu)+\frac{1}{2!}\left(\frac{2i}{I}\right)^2\left[\sum_\mu(\mathcal{E}_\mu-E_0-\frac{1}{2}i\Gamma_\mu)\right]^2-\frac{1}{2!}\left(\frac{2i}{I}\right)^2\sum_\mu(\mathcal{E}_\mu-E_0-\frac{1}{2}i\Gamma_\mu)^2+\cdots}.$$

We further choose $E_0$ such that

$$E_0 = \frac{1}{N}\sum_{\mu=1}^{N}\mathcal{E}_\mu.$$

This allows us to write the above expression as

$$\langle U\rangle = \exp(-2i\varphi)\frac{1-\frac{1}{I}\sum_{\mu=1}^{N}\Gamma_\mu+\frac{1}{2!}\left(\frac{1}{I}\sum_\mu\Gamma_\mu\right)^2-\frac{1}{2!}\left(\frac{2i}{I}\right)^2\sum_\mu(\mathcal{E}_\mu-E_0+\frac{1}{2}i\Gamma_\mu)^2+\cdots}{1+\frac{1}{I}\sum_{\mu=1}^{N}\Gamma_\mu+\frac{1}{2!}\left(\frac{1}{I}\sum_\mu\Gamma_\mu\right)^2-\frac{1}{2!}\left(\frac{2i}{I}\right)^2\sum_\mu(\mathcal{E}_\mu-E_0-\frac{1}{2}i\Gamma_\mu)^2+\cdots}.$$

We now take the limit when $I$ and $N$ both become large, such that the ratio $I/N = 2\Delta E/\pi N = 2D/\pi$ remains constant, where $D$ is the mean spacing of the poles $z_\mu$. Introducing the average width $\langle\Gamma_\mu\rangle$ in the usual way,[2,6]

$$\frac{1}{I}\sum_\mu \Gamma_\mu = \tfrac{1}{2}\pi \frac{(1/N)\sum \Gamma_\mu}{(\Delta E/N)} = \tfrac{1}{2}\pi \frac{\langle\Gamma_\mu\rangle}{D},$$

and dropping terms of the order of $\leqslant 1/I$, we arrive at the following expression for $\langle U\rangle$:

$$\langle U\rangle = \exp(-2i\varphi - \pi\langle\Gamma_\mu\rangle/D). \tag{8}$$

It is easy to check that this expression reduces to the familiar expression[6]

$$\langle U\rangle = \exp(-2i\varphi)[1-\pi\langle\Gamma_\mu\rangle/D],$$

when $\langle\Gamma_\mu\rangle/D \ll 1$.

We next consider the case of $m$ open channels. For this case the matrix $G_\mu$ satisfies the following sum rule[3]:

$$\sum_{\mu=1}^{N} G_\mu = \sum_{\mu=1}^{N} Y_\mu, \tag{9}$$

where the elements of the real symmetric matrix $Y_\mu$ are defined in terms of a compound-nucleus Hamiltonian. The diagonal elements of $Y_\mu$ are related to the total width $\Gamma_\mu$ of the level $\mu$ in the following fashion:

$$\sum_{\mu=1}^{N}\sum_{c=1}^{m} Y_{\mu cc} = \sum_{\mu=1}^{N} \Gamma_\mu. \tag{10}$$

Denoting the ensemble average by $\langle\,\rangle$, we find from expression (10) that

$$\sum_{c=1}^{m} \langle Y_{\mu cc}\rangle = \langle\Gamma_\mu\rangle, \tag{11}$$

and therefore the average values of $Y_{\mu cc}$ summed over the channel index $c$ give the average total widths. In this sense $\langle Y_{\mu cc}\rangle$ or $\langle G_{\mu cc}\rangle$ can be taken as the average partial widths.

To derive an expression for the matrix $\langle U\rangle$ we follow the same procedure as was used for the purely elastic-scattering case. Instead of expression (6), we now get the matrix relation

$$\langle U\rangle = V \prod_{\mu=1}^{N} \frac{E_0 + \frac{1}{2}iI - \theta_\mu}{E_0 + \frac{1}{2}iI - z_\mu} V, \tag{12}$$

where the matrix $\theta_\mu$ satisfies the sum rule

$$\sum_{\mu=1}^{N} \theta_\mu = \sum_{\mu=1}^{N} (z_\mu + iG_\mu). \tag{13}$$

This sum rule can be easily derived from expression (1). Since $\langle Y_\mu\rangle$ is a positive-definite, real, symmetric matrix, a real orthogonal matrix $T$ can be used to diagonalize it:

$$\tilde{T}\langle Y_\mu\rangle T = A_d. \tag{14}$$

Using expressions (12)-(14), making the earlier choice for $E_0$, and doing the same limiting process as we had done for the elastic-scattering case, we arrive at the desired result

$$\langle U\rangle = VT\exp(-\pi A_d/D)\tilde{T}V. \tag{15}$$

We note that expression (15) is derived for the general case, in which we have not put any restrictions on the background matrix $V$ or the complex quantities $G_\mu$. For the purely elastic-scattering case, expression (15) reduces to expression (8) as it should. Using this expression we can easily get expressions for the transmission coefficients $T_c$:

$$T_c = 1 - |\langle U_{cc}\rangle|^2, \tag{16}$$

and the average partial total cross section $\langle\sigma_c^{\text{tot}}\rangle$,

$$\langle\sigma_c^{\text{tot}}\rangle = 2\pi\lambda\!\!\!^{-}{}_c^2[1-\mathrm{Re}\langle U_{cc}\rangle]. \tag{17}$$

We now pass a few remarks about the procedure which we have adopted here and a relation which has been obtained by Moldauer[1] to determine $\langle U\rangle$. This relation for $\langle U\rangle$ was then solved only for the case in which direct reactions were absent, which is not the case treated here. Our relations (6) and (12) are exact and have been obtained from a completely unitary matrix with no restrictions on the direct reactions. It is unitarity which allows us to express $G_\mu$ in terms of $z_\mu$ only, in the case of purely elastic scattering, and to obtain the sum rules for the matrix $G_\mu$ when we are dealing with an arbitrary number of open channels. Since these relations implied by unitarity are not used in Ref. 1, we cannot compare our most general result given by expression (15) with the approximate result given in Ref. 1.

For the purely elastic-scattering case, our procedure can also be used to find $\langle U^2\rangle$. We can then immediately get expressions for the average and the mean-square deviation of the elastic-scattering cross section $\sigma$. These are given by

$$\langle\sigma\rangle = (2\pi\lambda\!\!\!^{-2})[1-(\cos 2\varphi)\exp(-\pi\langle\Gamma_\mu\rangle/D)], \quad (18)$$

$$\langle\sigma^2\rangle - \langle\sigma\rangle^2 = 2(\pi\lambda\!\!\!^{-2})^2[1-\exp(-2\pi\langle\Gamma_\mu\rangle/D)]. \quad (19)$$

As expected, expressions (18) and (19) reduce to the usual expressions obtained in the approximation of $\langle\Gamma_\mu\rangle/D \ll 1$.

I would like to thank Professor Virendra Singh and Dr. N. D. Sen Gupta for helpful discussions.

[1]P. A. Moldauer, Phys. Rev. Letters 19, 1047 (1967).
[2]T. Ericson, Ann. Phys. (N.Y.) 23, 390 (1963).
[3]Nazakat Ullah and C. S. Warke, Phys. Rev. 164, 1316 (1968), and Phys. Letters 26B, 556 (1968).
[4]A. M. Lane and R. G. Thomas, Rev. Mod. Phys. 30, 257 (1958).
[5]H. Feshbach, Ann. Phys. (N.Y.) 5, 357 (1958), and 19, 287 (1962), and 43, 410 (1967).
[6]H. Feshbach, A. K. Kerman, and R. H. Lemmer, Ann. Phys. (N.Y.) 41, 230 (1967).

# AN IMPROVED MODEL FOR COSMIC-RAY PROPAGATION

Peter D. Noerdlinger
Department of Physics and Astronomy, University of Iowa, Iowa City, Iowa
(Received 1 April 1968)

A model for cosmic-ray propagation derived by Jokipii is modified to take into account particle mirroring. The diffusion coefficient for particles of known velocity and rigidity is then determined by a limited portion of the field spectrum, instead of being affected by all frequencies above a limiting one.

The interplanetary magnetic field, at least up to about 1.2 A.U. from the sun, is known to be fairly ordered, lying principally at the garden-hose angle,[1,2] but with fluctuations of order 30% on quiet days and more on disturbed days. Jokipii analyzed cosmic-ray propagation in a field of this type, and showed[3,4] that the motion perpendicular to the mean field $\vec{B}_0$ is controlled by diffusion in pitch angle $\theta = \cos^{-1}\mu$. Following Jokipii[3] we denote the direction of $\vec{B}_0$ by $z$, the particle velocity by $V$, its charge by $Ze$, and its energy by $\gamma m_0c^2$. Define $\omega_0$ and $r_c$ by $\omega_0 = B_0eZ/\gamma m_0 c = V/r_c$. The power spectrum $P_{ij}(f)$ of the magnetic field fluctuation $\vec{B}_1 \equiv \vec{B} - \vec{B}_0$ is fairly isotropic in the $xy$ plane,[2,5,6] and we denote the part $P_{xx} = P_{yy}$ that affects diffusion along $\vec{B}_0$ by $P(f)$. This spectrum is attributed to a power spectrum $P(kV_w/2\pi)$ of field irregularities of wave number $k$ being carried past the spacecraft at the solar-wind velocity $V_w$.

Jokipii's theory, as originally set forth,[3] leads to divergent expressions if $P(f)$ falls off as steeply as $f^{-2}$ at large $f$. A later modification of Jokipii's avoids the divergence problem, but it depends on an assumption of near isotropy for the particles that we consider unnecessary,[7] and retains a dependence on the value of $P(f)$ for all $f$ exceeding $V_w\omega_0/2\pi V$. This dependence would not produce very incorrect results unless $P(f)$ were found to have unexpected sizable spikes at high frequency, but the theory to be presented here derives an upper limit to the frequencies that are significant for scattering particles of known rigidity. This is physically more satisfactory. Because we prefer[7] not to restrict the form of the particle distribution function unduly, we proceed from the original Jokipii theory[3] with the review of a few key equations, and then introduce the modification for mirroring.

Ignoring the slow diffusion in the $xy$ plane, and simplifying to a time-independent diffusion problem, we obtain from (J26)

$$2\mu V(\partial n/\partial z) = (\partial/\partial\mu)[\Delta(\partial n/\partial\mu)], \quad (1)$$

where we denote by $\Delta$ the Fokker-Planck coefficient for diffusion in $\mu$, $\Delta \equiv \langle(\Delta\mu)^2\rangle/\Delta t$. Denote by $N$ the particle density

$$N(z) = \int_{-1}^{1} n(\mu, z)\,d\mu. \quad (2)$$

For a diffusion problem, we expect $N(z)$ to be linear in $z$, so we decompose $n(\mu, z)$ into an isotropic part $n_0(z)$ linear in $z$ and an anisotropic part $n_1$ independent of $z$:

$$n(\mu, z) = \tfrac{1}{2}N_0(1+\alpha z) + n_1(\mu) \equiv n_0(z) + n_1(\mu). \quad (3)$$

Thus, $\partial N/\partial z = N_0\alpha$, and the diffusion coefficient $D_{zz}$ is defined by

$$D_{zz} = -F/N_0\alpha, \quad (4)$$