MAGNETOPLASMA MODES IN ALKALI METALS*

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It is shown that Fermi-liquid interactions in the alkali metals lead to a family of long-wavelength electromagnetic modes propagating parallel to a uniform magnetic field, near the cyclotron frequency. In contrast to the perpendicular case, however, there is a cutoff wavelength that increases with decreasing Fermi-liquid interaction, and can be in the range of 10 to 100 cyclotron radii, for the size of interaction now indicated in potassium.

Platzman and Walsh[1-3] have measured the dispersion relation of some of the plasma modes propagating perpendicular to an applied magnetic field in potassium, and have emphasized that such experiments enable one to evaluate the spin-independent Fermi-liquid parameters of the electronic system in alkali metals. A general theoretical discussion of the dispersion relation for plasma modes propagating at an arbitrary angle to the magnetic field was given by Silin.[4] More recently two of us[5] have reformulated the problem of propagation perpendicular to the field in a form particularly suitable for wavelengths large compared with the cyclotron radius ($X = |kv_F/\omega_c| \ll 1$).

In this note we extend the approach of Ref. 5 to the case of propagation parallel to the field. We find that a family of long-wavelength plasma modes continues to exist at frequencies near the cyclotron frequency, but, in contrast to the perpendicular case where modes propagate at any wavelength, all modes now have a short-wavelength cutoff. Roughly speaking, modes will propagate only if X is less than a typical Fermi-liquid parameter, A. Very little is now known about the A's, but data indicate[3] that in potassium A_2 and A_3 are on the order of 10^{-2}. If the cutoff wavelength is really this large the modes will be substantially more difficult to see in the kind of standing-wave experiment by which the perpendicular propagating modes were observed,[1,2] thicker samples, higher purity, and stronger magnetic fields being required. However, because there is as yet no evidence that the small size of A_2 and A_3 in potassium is typical of the alkalis, and because the purities and field strengths necessary to observe the parallel propagating plasma modes need not be inaccessible, it seems important that experimentalists be aware of the theoretical prediction that the modes exist.[6]

The frequencies of the modes propagating parallel to a uniform magnetic field in the collisionless regime ($\omega\tau \gg 1$) are given by the eigenfrequencies of the kinetic equation

$$\omega\nu - kv_z\bar{\nu} - \omega_c \frac{1}{i}\frac{\partial}{\partial\varphi}\bar{\nu} = \frac{3\omega_p^{*2}}{\omega}\left[\hat{v}_z \cdot \langle \hat{v}_z \bar{\nu}\rangle + \frac{\omega^2}{\omega^2 - k^2c^2} \times (\hat{v}_x \cdot \langle \hat{v}_x \bar{\nu}\rangle + \hat{v}_y \cdot \langle \hat{v}_y \bar{\nu}\rangle)\right]. \quad (1)$$

The magnetic field is taken to be along the z axis, and $\nu(\theta,\varphi)$ describes the deformation of the Fermi surface associated with the mode. If

$$\nu = \sum \nu_{lm} Y_{lm}(\theta,\varphi), \quad \bar{\nu} = \sum \bar{\nu}_{lm} Y_{lm}(\theta,\varphi), \quad (2)$$

then

$$\bar{\nu}_{lm} = \gamma_l \nu_{lm}, \quad \gamma_l = 1 + A_l = 1 + F_l/(2l+1), \quad (3)$$

where the F_l or A_l are the spin-independent Fermi-liquid interaction parameters.[7] Except for the direction of $\vec{k}$ and the absence of an explicit external driving field, (1) is identical to Eq. (1) of Ref. 5, where a fuller explanation of the notation is given.

Using the method of Ref. 5 one easily calculates that to second order in k, normal modes propagate at frequencies

$$\omega_{lm} = m\omega_c\gamma_l\left(1 + \frac{X^2}{m^2}\left[\frac{l^2-m^2}{4l^2-1}\frac{\gamma_{l-1}}{\gamma_l - \gamma_{l-1}} + \frac{(l+1)^2-m^2}{4(l+1)^2-1}\frac{\gamma_{l+1}}{\gamma_l - \gamma_{l+1}}\right]\right), \quad l \geq m,\ l > 2; \quad (4)$$

$$\omega_{2m} = m\omega_c\gamma_2\left[1 + \frac{X^2}{m^2}\left(\frac{9-m^2}{35}\frac{\gamma_3}{\gamma_2 - \gamma_3}\right)\right].$$

The coefficients of the quadratic terms in (4) all diverge in the limit of vanishing Fermi-liquid interaction ($\gamma_l \to 1$). This is a reflection of the fact that when $\gamma_l \equiv 1$ Eq. (1) has a continuous spectrum (at $|\omega \pm m\omega_c| \leq kv_F$) rather than a discrete one, no matter how long the wavelength,

while the derivation of (4) assumes a discrete spectrum of excitations.[8]

This suggests that as the Fermi-liquid interaction is turned off, the region above $X=0$ for which the spectrum of normal modes is discrete and (4) is valid shrinks down to a point. Putting it more physically, one suspects that as kv_F/ω_C decreases for fixed nonvanishing Fermi-liquid interaction, successively more of the propagating discrete modes (4) will separate off from the continuum.

This conjecture is confirmed and illustrated by explicit solutions to the kinetic equation in which only a finite number of the A_l are taken to be nonzero.[9] First note that (1) separates into independent equations for modes with azimuthal dependence $e^{im\varphi}$. Since only the $m=1$ modes will be coupled to an electromagnetic probe we consider this case alone.[10] As in Ref. 5 we consider (1) to be an equation in a space spanned by the spherical harmonics, and note that for modes with frequencies near ω_C, provided $\omega_p^2 \gg \omega_C^2 - k^2c^2$, the effect of the term in square brackets is to require that the solution have negligible projection in the $l=1$ subspace. We can therefore drop the electromagnetic term[11] provided we project the resulting equation onto the space with $l \geq 2$:

$$P(\omega P\nu - kv_z P\bar{\nu} - \omega_C P\bar{\nu}) = 0, \tag{5}$$

where P projects on the space $m=1, l \geq 2$.

If we expand the solution to (5) as

$$\nu = \sum_{l=2}^{\infty} \bar{\nu}_l Y_{l1}, \tag{6}$$

then (5) reduces to

$$\bar{\nu}_l = \left(z+\frac{1}{X}\right)\sum_{l'=2}^{\infty} K_{ll'}C_{l'}\bar{\nu}_{l'} - \frac{1}{\sqrt{5}}\bar{\nu}_2 K_{l1}, \quad l \geq 2; \tag{7}$$

where

$$z = \frac{\omega-\omega_C}{kv_F}, \quad C_l = \frac{A_l}{1+A_l}, \tag{8}$$

and

$$K_{ll'} = \int \frac{Y_{l1}^* Y_{l'1}}{z-\cos\theta} d\Omega. \tag{9}$$

[The entire effect of the projection operators is to produce the last term in (7).]

If only $C_2, C_3, \cdots, C_{n+1}$ are nonzero, then for $l=2, \cdots, n+1$, (7) becomes an $n \times n$ homogeneous system. Setting the determinant of the coefficients to zero gives n functions, $X_l(z)$, $l=2, \cdots, n+1$. One easily reads off the dispersion relations for n discrete modes from these functions, by plotting $zX_l(z) = (\omega_{l1}-\omega_C)/\omega_C$ vs $X_l(z) = kv_F/\omega_C$ for positive values of $X_l(z)$ [which is an odd function of z because of the symmetry $K_{ll'}(-z) = (-1)^{l+l'+1}K_{ll'}(z)$]. The long-wavelength behavior of the modes comes from values of $|z| \gg 1$.[12] The mode merges into the continuum at $|z|=1$, so that the cutoff wave vector for the mode ω_{l1} is $k_l = \omega_C |X_l(1)|/v_F$.

Keeping, for example, only C_2 we find the root

$$X_2(z) = A_2/z[1-\gamma_2 g(z)], \tag{10}$$

where

$$g(z) = \frac{(5z^2-1)K_{11}(z)-5z}{5z[zK_{11}(z)-1]}, \tag{11}$$

and

$$K_{11}(z) = \frac{3}{2}z\int_0^1 \frac{du(1-u^2)}{z^2-u^2}. \tag{12}$$

The function $g(z)$ decreases monotonically from $\frac{2}{5}$ at $z=1$ to 0 at $z=\infty$. Therefore, provided $1+A_2 < \frac{5}{2}$ (i.e., $F_2 < 7.5$), there is a single root[13] that cuts off at $kv_F/\omega_C = |X_2(1)| = |5A_2/(3-2A_2)|$. The dispersion relation for this mode from zero k up to the cutoff is shown in Fig. 1 for the value $A_2 = -0.03$ suggested by Platzman, Walsh, and Foo[3] in potassium.

The presence of only a single branch is, of course, due to the neglect of the higher C_l's. Keeping both C_2 and C_3 indicates the more general structure as well as illustrating the sensitivity to C_3 of the $l=2$ root. In this case the two

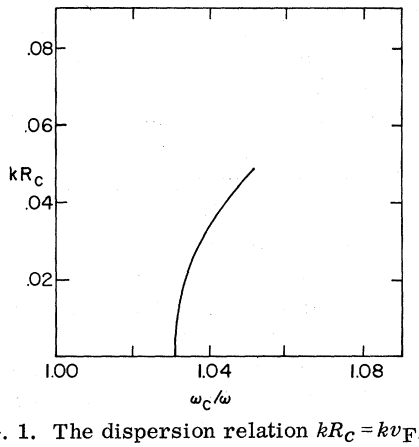

FIG. 1. The dispersion relation $kR_C = kv_F/\omega_C$ vs ω_C/ω, for $A_2 = -0.03$, $A_l = 0$, $l > 2$. The curve terminates at the cutoff wave vector, $kR_C = 0.049$.

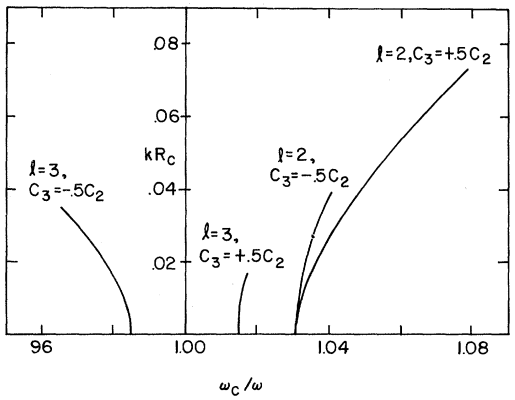

FIG. 2. The dispersion relation $kR_C = kv_F/\omega_C$ vs ω_C/ω, for the two modes present when $A_l = 0, l > 3$. A_2 is -0.03 as in Fig. 1. Curves are drawn for the two choices $C_3 = \pm 0.5\, C_2$, which have no particular significance except as probable limits to the range over which A_3 is likely to vary. The curves terminate at their cutoff wave vectors.

branches are most conveniently expressed by

$$X_2(z) = \frac{A_2/z}{1-\gamma_2[1-Y_2(z)]}, \tag{13}$$

$$X_3(z) = \frac{A_3/z}{1-\gamma_3[1-C_3 Y_3(z)/C_2]}, \tag{14}$$

where Y_2 and Y_3 are the two roots of

$$(1-Y)(1-35z^2C_3gY/8C_2) = g \tag{15}$$

satisfying

$$\lim_{z\to\infty} Y_2 = 1, \quad \lim_{z\to\infty} Y_3 = C_2/C_3. \tag{16}$$

Both branches are shown in Fig. 2 for two values of A_3 ($C_3/C_2 = \pm\frac{1}{2}$) and $A_2 = -0.03$. Evidently the $l = 2$ mode becomes more sensitive to A_3 as the cutoff is approached.

Further numerical analysis seems pointless, except for a careful study of the actual boundary-value problem for transmission through a slab, until some experimental data become available, and Figs. 1 and 2 should be regarded as typical rough characterizations of the structure of two of the modes one might find. Although experimental considerations will favor seeking the modes at the largest values of kv_F/ω_C (unless the values of the Fermi-liquid parameters in potassium turn out to be atypically low), one should bear in mind that as the cutoff wave vector is approached, the coupling of the modes to an electromagnetic probe will become exceedingly small. It is therefore likely that only wave vectors rather below the cutoff will be observed.

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[1]P. M. Platzman and W. M. Walsh, Jr., Phys. Rev. Letters 19, 514 (1967), and 20, 89 (1968).

[2]W. M. Walsh, Jr., and P. M. Platzman, Phys. Rev. Letters 15, 784 (1965).

[3]P. M. Platzman, W. M. Walsh, Jr., and E-Ni Foo, to be published.

[4]V. P. Silin, Zh. Eksperim. i Teor. Fiz. 35, 1243 (1958) [translation: Soviet Phys.—JETP 8, 870 (1959)].

[5]N. D. Mermin and Y. C. Cheng, Phys. Rev. Letters 20, 839 (1968).

[6]One of us (N.D.M.) has described the structure of these modes in a post-deadline talk in a meeting of the American Physical Society, Berkeley, 19 March 1968 (unpublished). D. Fredkin and collaborators, and P. Nozières have independently drawn conclusions similar to those presented here (private communications), but no experimental evidence for (or against) the modes has yet been produced.

[7]The F_l notation is Landau's and is used in most of the helium literature. Silin writes A_l instead of F_l, while Platzman and Walsh use $A_l = F_l/(2l+1)$, which seems to be catching on in the plasma literature.

[8]This is in sharp contrast to the case of perpendicular propagation, in which the perturbation results are well behaved in the free electron limit and the spectrum remains discrete (though highly degenerate).

[9]While the truncation of A still leaves one with a nasty computational problem in the case of perpendicular propagation, in the parallel case it reduces the computation to finding the roots of a polynomial.

[10]Silin discusses the conceptually more straightforward but experimentally less accessible $m = 2$ modes in Ref. 4.

[11]This approximation is equivalent to the Platzman-Walsh approximation of finding the dispersion relation from the condition that the mode carry zero current.

[12]This is obvious from a comparison of the long-wavelength limit of (4) for ω_{l1} with the definition $z = (\omega-\omega_C)/kv_F$.

[13]For what appear to be unrealistically large values of the Fermi-liquid parameters, the structure of the spectrum can be considerably more complicated. In the F_2-only model, for example, the cutoff recedes to infinite k at $F_2 = 7.5$ and at least one mode (and in some regions, two) propagate at any k for $F_2 > 7.5$.