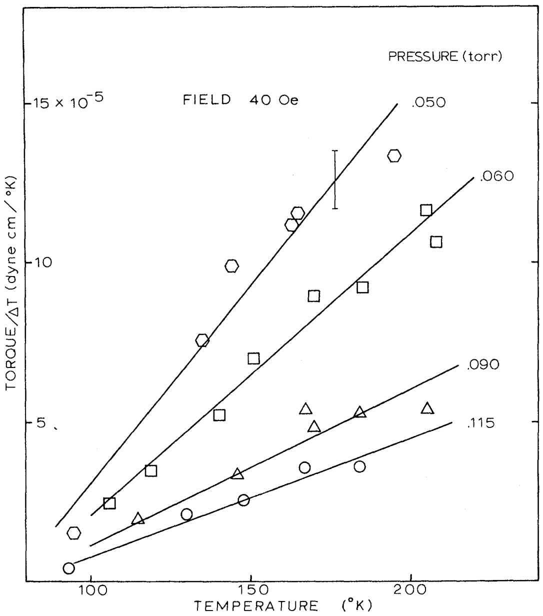

FIG. 2. Temperature dependence of Scott effect for nitrogen gas from 100 to 200°K. The typical error bar shown is large due to inaccuracies in ΔT measurements.

clude that the torque is larger for a larger mean free path. This means that a transverse force acts on the molecule for a longer time between collisions in order to give a transverse momentum. It should be pointed out that the lines through the data points appear to extrapolate to zero near the liquefaction temperature for the nitrogen. The dimensions of the apparatus were such that the mean free path of the N_2 gas was small compared to the distance from the torsion pendulum to the container wall.

The error bar shown in Fig. 2 is rather large because of inaccuracies in the measurement of the temperature gradient. Our continuing effort will be to reduce this error by reducing the vertical temperature gradient.

The observed torque, now called the Scott effect, must be caused by a transverse linear momentum transport and appears to be related to the Senftleben effect[3] which is a change of heat conductivity by the polyatomic gas in a magnetic field. Knaap and Beenakker[4] have published a phenomenological theory for the Senftleben effect which shows that there will be energy and momentum transport perpendicular to the direction of a magnetic field and perpendicular to the direction of a temperature gradient in polyatomic gases. Levi and Beenakker[5] have applied the theory to the new torque effect and have shown agreement with the room-temperature measurements of Scott, Sturner, and Williamson. They do not discuss the temperature dependence of the observed torque. It should be an interesting theoretical problem to calculate from first principles how the lack of spherical symmetry of N_2 gas can cause a weak magnetic field to give rise to a transverse force on the molecule.

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[1]G. G. Scott, H. W. Sturner, and R. M. Williamson, Phys. Rev. 158, 117 (1967).

[2]M. R. Baker, C. H. Anderson, and N. F. Ramsey, Phys. Rev. 113, A1533 (1964).

[3]H. Senftleben, Z. Physik 31, 961 (1930).

[4]H. F. P. Knaap and J. J. M. Beenakker, Physica 33, 543, 503 (1967).

[5]A. C. Levi and J. J. M. Beenakker, Phys. Letters 25, 350 (1967).

BEHAVIOR OF THE CORRELATION FUNCTION NEAR THE CRITICAL POINT

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One of the more important aspects of the problem of cooperative transitions is the relationship between the correlation function and the thermodynamic properties. It is our main purpose in this note to point out that there is a general relation limiting the possible distance dependence of the correlation function at the critical point. Let the correlation function there decrease for large distance r as r^{-n}. In terms of the exponent δ characterizing the critical isotherm ($H \sim m^{\delta}$, at $T=T_c$) we show that there is a number $\hat{n}=2d/(\delta+1)$ for a system in d-dimensional space such that $n \geq \hat{n}$. Among other consequences, this has the result that in three dimensions the Ornstein-Zernike theory result ($n=d-2$) at the critical point cannot be correct unless $\delta \geq 5$ and thus could not apply to a system possessing the "classical" or van der Waals thermodynamic behavior.

Our results[1] are valid for the order-parameter–order-parameter correlation function for any system, including the gas-liquid critical point and the Bose transition of a quantum system, but for simplicity we use language appropriate to the ferromagnetic Curie transition. In this case the order parameter is the mean magnetization $\bar{m}$ per particle, and the conjugate intensive parameter the magnetic field H. In the neighborhood of the critical point the deviations of the value of the dynamical variable m from its mean value $\bar{m}$ are correlated over large distances. They can be described by the correlation function $C(r,t,\bar{m})$ [where $T=(1+t)T_c$ and T_c is the critical temperature], whose second moment is the square of a correlation length $R_c=R_c(t,\bar{m})$. This correlation length diverges at the critical point like $t^{-\nu}$ for $t\to+0$ [or $(-t)^{-\nu'}$ for $t\to-0$] at $H=0$, and like $H^{-\kappa}$ or $\bar{m}^{-\kappa\delta}$ at $t=0$. At the critical point, $C(r,0,0)$ decreases with r asymptotically like r^{-n}. The integral of the correlation function over a large but finite part (of linear dimension R) of a very large system is just the mean-square fluctuation of $\bar{m}$ in this part. Expressing this per particle, we have

$$\langle(m-\bar{m})^2\rangle_R\int_R d\vec{r}=\int_R C(r,t,\bar{m})d\vec{r}. \qquad (1)$$

In the limit, as R becomes infinite (letting the volume of the larger system tend to infinity first), the right-hand side of (1) is equal to the isothermal susceptibility $\chi=(\partial\bar{m}/\partial H)_T$. Noting that $C(r,t,\bar{m})$ is non-negative, and ignoring numerical factors, we can write

$$R^d\langle(m-\bar{m})^2\rangle_R=\int_R C(r,t,\bar{m})d\vec{r}\leq\chi(t,\bar{m}). \qquad (2)$$

At the critical temperature, therefore,

$$R^d\langle(m-\bar{m})^2\rangle_R\leq\chi(0,\bar{m})=\bar{m}^{-(\delta-1)}. \qquad (3)$$

Introducing σ_R as the fluctuation at the critical point, (1) gives

$$\sigma_R^{\ 2}\equiv\langle(m-\bar{m})^2\rangle_{R;\,t=0,\,\bar{m}=0}=R^{-n}. \qquad (4)$$

We now consider the fluctuations in a thermodynamic state for which the mean magnetization (for the total system, of course) differs from zero by a small amount proportional to σ_R itself (still at $t=0$), say $\bar{m}=\epsilon\sigma_R$, with $\epsilon\ll1$, but independent of R. We note that the fluctuation in region R can change by at most a correspondingly small amount, i.e.,

$$\langle(m-\bar{m})^2\rangle_{R;\,t=0,\,\bar{m}=\epsilon\sigma}=A_\epsilon\sigma_R^{\ 2}, \qquad (5)$$

where[2] A_ϵ is a number remaining near unity for small ϵ. From Eqs. (3) and (5) we now obtain the inequality

$$R^d\sigma_R^{\ 2}\leq(\epsilon\sigma_R)^{-(\delta-1)},$$

which with the use of (4) leads, as $R\to\infty$, to the desired result

$$n\geq2d/(\delta+1)=\hat{n}. \qquad (6)$$

A nonrigorous, but more physical, way of considering the origin of the inequality given by Eq. (6) is to note that the probability for the occurrence of a fluctuation m in the magnetization in region R is proportional to $\exp(-\Delta F/kT)$, where $\Delta F=\Delta F(R;T_c,m)$ is the extra free energy required by the constraint that the magnetization in R is m. We know that as R tends to infinity, ΔF tends to the limiting value for the infinite system, i.e., to $R^dm^{\delta+1}$. Suppose that for finite R one can represent this free energy as, say,

$$\Delta F=R^dm^{\delta+1}+(R^dm^b)^{1-a},$$

where the first term is the thermodynamic value and the second, with $a>0$, represents a finite-size contribution. It is easy to verify that, in the limit $R\to\infty$, the second term does not affect the distribution of fluctuations if $b>\delta+1$. If, however, $b<\delta+1$, that term dominates. From a calculation of the mean-square fluctuations and the use of Eq. (4) it follows that in the first case $n=\hat{n}$, but in the latter $n>\hat{n}$. Thus the presence of the finite-size term can reduce but not increase the size of the fluctuations. It is apparent from this point of view that one would normally expect the equality $n=\hat{n}$ (as in the case for the two-dimensional Ising model if δ equals the expected[3] value 15, since[4] $n=\frac{1}{4}$). However, the equality cannot be true in general since the ideal Bose gas provides a counter example[5]: For $d\leq4$, $n=\hat{n}$, but for $d>4$, $n>\hat{n}$. This latter situation is associated with a "sticking" of the thermodynamic exponents at their classical values for $d>4$ and in this sense is not "normal": The classical values arise from terms in the free energy possessing a Taylor expansion, the remaining singular part contributing only to higher derivatives. For $d \leq 4$ it is the singular part which dominates, leading to "normal" behavior and, in fact, the equality. This example incidentally is also suggestive concerning the origin of other inequalities possibly occurring in more general systems.

Away from the critical point (i.e., either $|t| > 0$, or $|\bar{m}| > 0$, or both, but $|t| \ll 1$, etc.) the decay of $C(r, t, \bar{m})$ should be dominated for large enough r (at fixed t and $\bar{m}$) by a convergence factor, $\exp(-r/R_c)$, where $R_c = R_c(t, \bar{m})$ is the correlation length.[6] Thus

$$\chi(t, \bar{m}) = \int_{R\to\infty} C(r, t, \bar{m}) d\vec{r} \cong \int_{R \leq R_c} C(r, t, \bar{m}) d\vec{r}.$$

One can now obtain further inequalities governing the behavior of the correlations by assuming the expected property that at a given r the correlation function decreases on moving away from the critical point, i.e., $C(r, t, \bar{m}) \leq C(r, 0, 0)$. Then

$$\chi(t, \bar{m}) \leq \int_{R_c} C(r, 0, 0) d\vec{r} = R_c^{\,d-n}. \tag{7}$$

It is then a straightforward matter to show that the correlation-length exponents satisfy the inequalities

$$\gamma \leq \nu(d-n) \leq d\nu(\delta-1)/(\delta+1);$$

$$\gamma' \leq \nu'(d-n) \leq d\nu'(\delta-1)/(\delta+1), \tag{8}$$

$$1 + 1/\delta \leq (d-n)(\delta+1)/(\delta-1) \leq d\kappa, \tag{9}$$

where $\chi(t, 0) = t^{-\gamma}$, etc.

The above arguments can also be extended to the case of the energy-energy correlation function $C_E(r, t, \bar{m})$, whose volume integral is

$$R^d \langle (\Delta E)^2 \rangle_R = \int_R C_E(r, t, \bar{m}) d\vec{r},$$

where the right-hand side equals the specific heat $c_{\bar{m}}$ in the limit as R becomes infinite. One then finds that (in an obvious extension of the notation)

$$n_E \geq 2d(1-\alpha)/(2-\alpha),$$

$$d\nu_E \geq 2-\alpha, \quad d\nu_E' \geq 2-\alpha',$$

$$d\kappa_E \geq a(2-\alpha)/\alpha, \tag{10}$$

where $c_{\bar{m}}$ behaves like $t^{-\alpha}$ [or $(-t)^{-\alpha'}$] and H^{-a}, for $H = 0$ and $t = 0$, respectively.

To discuss the relevance of the above inequalities to investigations of cooperative transitions, we first note from Eq. (6) that the Ornstein-Zernike (O.Z.) theory,[7] which predicts $n = d-2$, cannot be satisfied for all values of d and δ. For example, for both the O.Z. theory to be valid and for a system to have the classical value $\delta = 3$ requires $d \geq 4$. Alternatively, for the O.Z. theory to be valid in three dimensions requires $\delta \geq 5$. That the O.Z. and classical theories of the critical region are not in general mutually consistent is not surprising, since only the former depends explicitly on the dimensionality. It is interesting to note that present numerical evidence[3] for the three-dimensional Ising model suggests $\delta = 5$, as required for the validity of the O.Z. theory if (6) were obeyed as an equality. The numerical evidence[6] that n is approximately equal to 1.06 would, however, require the inequality. The O.Z. theory also gives[6] $\gamma = 2\nu$, thus satisfying the first equality in (8). If we make the reasonable supposition that the correlation lengths in $C_E(r, t, \bar{m})$ and $C(r, t, \bar{m})$ are the same, then $\nu_E = \nu$, and Eq. (10) then gives

$$\alpha \geq 2 - \tfrac{1}{2} d\gamma. \tag{11}$$

This relation has a bearing on a number of examples of three-dimensional systems, including a result often appearing in the literature,[8] namely that the O.Z. theory and $\gamma = 1$ result in $\alpha = \frac{1}{2}$; this corresponds to the equality in (11). The equality would also give, for $\gamma = \frac{4}{3}$ (Heisenberg model), $\alpha = 0$, and for $\gamma = \frac{5}{4}$ (Ising model), $\alpha = \frac{1}{8}$. If three-dimensional systems do obey these postulates and the thermodynamic equalities, there is only one free parameter, say γ. Thus,

$$\delta = 5, \quad \alpha = 2 - \tfrac{3}{2}\gamma, \quad \beta = \tfrac{1}{4}\gamma.$$

It is possible to draw other conclusions from the inequalities, such as the fact that $n \geq \frac{1}{2}$ for the one-dimensional model of Hemmer, Kac, and Uhlenbeck.[9] However, we conclude by simply noting that the inequalities derived here do not depend on any assumptions of homogeneity either for the thermodynamic functions or for the correlation function. In fact, it is known that for the ideal Bose gas the correlation function is homogeneous (as, in most dimensions, are the thermodynamic functions) of the form $r^{-n} f(r/R_c)$.[5] Only for $d \leq 4$ does the equality in (6) hold, however, as noted above. Homogeneity is not sufficient to guarantee the equalities.

[1]Some of these results were presented by M. J. Buckingham at the Seventh Summer Research Institute of the Australian Mathematical Society, Canberra, Australia, January, 1967 (unpublished). Some are also discussed by J. D. Gunton, thesis, Stanford University, 1966 (unpublished).

[2]This result, which is physically rather obvious, can be established formally by considering two concentric parts in a very large system. It is easily seen that, even by considering $A_\epsilon \ll 1$ for the larger part, this could not also be true for the smaller, resulting in a contradiction. (The other case $A_\epsilon \gg 1$ would only reinforce the inequality.) In the extreme case of a two-δ-function distribution corresponding to two-phase equilibrium, A_ϵ in (5) is $1-\epsilon^2$; for a Gaussian distribution $A_\epsilon = 1$.

[3]D. S. Gaunt, Proc. Phys. Soc. (London) 92, (1967), where other references are also given.

[4]B. Kaufman and L. Onsager, Phys. Rev. 76, 1244 (1949).

[5]J. D. Gunton and M. J. Buckingham, Phys. Rev. (to be published).

[6]M. E. Fisher, J. Math. Phys. 5, 944 (1964); J. Appl. Phys. 38, 981 (1967).

[7]L. S. Ornstein and F. Zernike, Proc. Acad. Sci. Amsterdam 17, 793 (1914); M. E. Fisher, Physica 28, 172 (1962).

[8]For example, see T. R. Choy and J. E. Mayer, J. Chem. Phys. 46, 110 (1967).

[9]P. C. Hemmer, M. Kac, and G. E. Uhlenbeck, J. Math. Phys. 4, 216 (1963).

PRESSURE ANISOTROPY AS A RESULT OF MAGNETIC BREMSSTRAHLUNG*

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In an isotropic distribution of particles undergoing magnetic bremsstrahlung, the particles moving at the largest angles to $\vec{B}$ lose energy the fastest. The resulting pressure anisotropy can lead to plasma instability. The particle velocities tend secularly to align with $\vec{B}$, slightly enhancing the effect.

Previous studies[1,2] of the behavior of distributions of particles undergoing magnetic bremsstrahlung ("synchrotron radiation") have concentrated on energy spectra, with little regard to angular distributions. In the ultrarelativistic approximation, the dependence of the nondimensionalized energy $\gamma = E/mc^2$ of such particles on time is given by

$$-d\gamma/dt = A\gamma^2 \sin^2\theta, \tag{1}$$

where θ is the angle between the velocity $\vec{v} = \vec{\beta}c$ and the magnetic field $\vec{B}$, and where

$$A = 2e^4B^2/3m^3c^5. \tag{2}$$

Commonly, $B\sin\theta$ is denoted by $B_\perp$, since it is the component of $\vec{B}$ perpendicular to $\vec{v}$. It is clear from (1) that for given initial energy γ, the particles with the largest θ will lose energy the fastest. If the initial distribution of particles or the injection spectrum is isotropic in θ, the resulting spectrum will exhibit an excess of pressure along $\vec{B}$. In the case of initial injection followed by decay, as would be appropriate in supernovae and possibly in catastrophic events in quasistellar objects, the time for development of significant anisotropy is comparable with that for significant energy loss. The most useful measure of anisotropy is the difference between the pressure along $\vec{B}$,

$$P_{\|} = 2\pi \int_{-1}^{1} d\mu \int_0^\infty dp \, \frac{\mu^2 p^4 f(p,\mu)}{m\gamma}, \tag{3}$$

and that orthogonal to $\vec{B}$,

$$P_\perp = \pi \int_{-1}^{1} d\mu \int_0^\infty dp \, \frac{(1-\mu^2)p^4 f(p,\mu)}{m\gamma}, \tag{4}$$

where f is the distribution of particles in ($\vec{x}$, $\vec{p}$) configuration space, supposed to be spatially uniform, and $\mu \equiv \cos\theta$. As an example, consider the initial differential energy spectrum

$$N(E)dE = n(\alpha-1)E_m^{\alpha-1}E^{-\alpha}dE, \quad E > E_m,$$
$$= 0 \text{ otherwise}, \tag{5}$$

where α is chosen greater than 2 to make (3) and (4) converge, and where n is the number density. In the present (ultrarelativistic) approximation, $N(E)$ is easily converted to a momentum distribution $f_0(p)$ normalized to $\int f_0(p)d^3p = n$ by use of the relation $E \sim pc$. (The symbol $\sim$ will denote equality under the ultrarelativis-