sum starts at zero and grows negative as a function of s. For s chosen not far above the $\Delta(1238)$, Gatto[6] notes that the Regge term is small compared with the nucleon or $\Delta(1238)$, and so he neglects it. His sum rule is a superconvergence relation for $B^{(+)}$ with the integral cut off at s. Thus it looks like a FESR except that the Regge sum is set equal to zero. Experimentally, this is much better than the FESR with just the P and P', although it is far from perfect if one uses phase shifts[7] instead of the narrow-width resonance approximation.

This example shows that the FESR are not universally valid, even at the upper limit of present phase-shift analyses, and contains a lesson about models for fitting the experimental data below the Regge region. The criticism based on the FESR against the interference model (the amplitude parametrized as a sum of crossed-channel Regge poles plus a sum of direct-channel resonances) is valid only if the FESR are well satisfied. In fact, Gatto's sum rule suggests the interference model as correct for the $B^{(+)}$ amplitude. However, the accuracy of the FESR for $B^{(-)}$ with just the ρ trajectory would suggest the Dolen-Horn-Schmid[2] prescription for parametrizing the $B^{(-)}$ amplitude. (They parametrize the amplitude as a sum of the crossed-channel Regge poles plus direct-channel resonances minus the average of the resonances.) Neither parametrization is adequate to describe all of πN scattering below the Regge region. Chiu and Stirling[8] have already shown that the interference model is inadequate. It is unwise to assume that Regge asymptotic behavior is more accurate at low energies when expressed in terms of the FESR than when expressed directly in terms of scattering amplitudes.

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IMPOSSIBILITY OF FINITE CHARGE RENORMALIZATION IN SIXTH ORDER*

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Making use of a recent result obtained by Jackiw in the context of the renormalization group we examine the question of whether there exists the possibility of obtaining finite charge renormalization by means of an eigenvalue condition on the bare coupling constant. It is shown that to sixth order in e_0 a cancellation of the divergences encountered in perturbation theory cannot occur.

Quantum electrodynamics has long enjoyed a relatively privileged position in the study of relativistic field theories. Although this is in large part a consequence of the impressive successes which it has achieved in predicting experimentally observable quantities, its usefulness has been further enhanced by the fact that the divergences of this theory are at least partially understood, a circumstance which has suggested that it might well be the most promising candidate for a completely finite field theory. In particular, the freedom which one has available in the choice of gauge is known to imply the possibility of eliminating the divergences in Z_2 (the electron wave-function renormalization) while the Ward identity implies $Z_1=Z_2$ and the consequent finiteness of the vertex renormalization in the same gauge. In addition, the problem of the photon mass has been shown to resolve itself provided that sufficient care is taken to ensure the gauge invariance of the current operator.[1]

During the last several years a rather extensive discussion of electrodynamics by Baker, Johnson, and Willey[2,3] has attempted to dispose of the remaining divergences of quantum electrodynamics by arguing that nonperturbative methods can lead to a finite electron mass m provided that the electron bare mass is taken to be zero. If one accepts this result, there remains only the problem of disposing of the divergences encountered in the perturbation-theory calculation of the charge renormalization constant Z_3. It is found by Johnson, Willey, and Baker[3] that in a model in which the photon propagator is replaced by $1/k^2$ in all internal lines, Z_3^{-1} has the form

$$Z_3^{-1} = 1 + f(\alpha_0)\ln(\Lambda^2/m^2) + \text{finite terms},$$

where Λ is a cutoff mass and α_0 is the unrenormalized fine-structure constant $e_0^2/8\pi^2$. Consequently these authors assert that the theory they consider can be finite if there exists a zero of $f(\alpha_0)$ for some positive α_0. The function $f(\alpha_0)$ has been calculated to fourth order by Jost and Luttinger[4] and to sixth order by Rosner[5] with the result

$$f(\alpha_0) = \tfrac{2}{3}\alpha_0 + \alpha_0^2 - \tfrac{1}{4}\alpha_0^3. \qquad (1)$$

On the basis of the negative sign for the sixth-order term it has been argued by Johnson et al. that there is some basis to anticipate the existence of a solution of the eigenvalue condition

$$f(\alpha_0) = 0$$

for some $\alpha_0 > 0$. Thus if the model proposed by these authors can be considered to be a reasonable facsimile of electrodynamics, one might well be tempted to accept this result as providing a basis for a completely finite theory of the interacting electromagnetic field.

More recently, however, this program has been criticized by Jackiw.[6] Using results obtained by Gell-Mann and Low[7] in the context of the renormalization group, he shows that Z_3^{-1} has the form

$$Z_3^{-1} = \frac{\alpha_0}{Q(\alpha_0)} + \alpha_0 \sum_{n=1} \frac{\ln^n(m^2/\Lambda^2)}{n!} \times \left[\alpha_0 \varphi(\alpha_0)\frac{d}{d\alpha_0}\right]^n Q^{-1}(\alpha_0), \qquad (2)$$

where $\varphi(\alpha_0)$ and $Q^{-1}(\alpha_0)$ are functions of the bare coupling constant which can be calculated in perturbation theory. In particular, these functions can be written to sixth order (using the notation of Jackiw) as

$$\varphi(\alpha_0) = a_1\alpha_0 + b_1\alpha_0^2 + (c_1 - a_0 b_1)\alpha_0^3,$$

$$Q^{-1}(\alpha_0) = 1/\alpha_0 + a_0 + b_0\alpha_0 + (c_0 - a_0 b_0)\alpha_0^2,$$

where the constants a_i, b_i, and c_i are defined by the asymptotic form of the renormalized photon propagator

$$D^{-1}(k^2 \to \infty) = k^2[1 - \alpha A - \alpha^2 B - \alpha^3 C]$$

with

$$A = a_1 \ln(k^2/m^2) + a_0 + O(1/k^2),$$

$$B = b_1 \ln(k^2/m^2) + b_0 + O(1/k^2),$$

$$C = c_2 \ln^2(k^2/m^2) + c_1 \ln(k^2/m^2) + c_0 + O(1/k^2).$$

It is to be noted that although all terms of the form $\ln(m^2/\Lambda^2)$ can be made to vanish if α_0 can be chosen to be a zero of $\varphi(\alpha_0)$, it is not sufficient to require only that the coefficient of the $\ln(m^2/\Lambda^2)$ term be zero in order to obtain a finite value of Z_3. Clearly a value of α_0 which corresponds to a zero of $(d/d\alpha_0)Q^{-1}(\alpha_0)$ can also cause the absence of the $\ln(m^2/\Lambda^2)$ term without necessarily implying that Z_3 be finite. Indeed, it is easy to see from (2) that Z_3 can be finite only (i) if $\varphi(\alpha_0) = 0$ or (ii) if $(d^n/d\alpha_0^n)Q^{-1}(\alpha_0) = 0$ for all $n \geq 1$. Since (ii) can obtain only for the case $Q(\alpha_0) = \text{const}$ (a possibility which is not consistent with perturbation theory), we conclude that according to the results of the renormalization group the only way in which one can hope to obtain a finite theory is to choose α_0 to be a zero of $\varphi(\alpha_0)$.

It has also been observed by Jackiw that the function $f(\alpha_0)$ of Johnson et al. is not $\varphi(\alpha_0)$ to sixth order but rather $\varphi(\alpha_0)(1 - \alpha_0^2 b_0)$; so using Eq. (1), one obtains to the same order

$$\varphi(\alpha_0) = \tfrac{2}{3}\alpha_0 + \alpha_0^2 + \alpha_0^3(-\tfrac{1}{4} + \tfrac{2}{3}b_0), \qquad (3)$$

and consequently no conclusion is possible concerning the sign of the α_0^3 term unless b_0 can be calculated. Although Jackiw claims that b_0 is not known, we show here that b_0 can in fact be obtained relatively simply from the complete fourth-order polarization operator calculated by Källén and Sabry.[8]

Writing the photon propagator as

$$D^{-1}(k^2) = k^2[1 + \Pi(k^2)]$$

and using the aforementioned result of Källén and Sabry, one finds the following expression for the fourth-order contribution to $\Pi(k^2)$ for $k^2 > 0$

and real:

$$\frac{\Pi(k^2)}{1+\Pi(k^2)}=\frac{4}{3}\alpha^2\Big\{-\frac{13}{108}+\frac{11}{72}\delta^2-\frac{1}{3}\delta^4+\delta\Big(\frac{19}{29}-\frac{55}{72}\delta^2+\frac{1}{3}\delta^4\Big)\ln\frac{1+\delta}{|1-\delta|}-\Big(\frac{33}{32}+\frac{23}{16}\delta^2-\frac{23}{32}\delta^4+\frac{\delta^6}{12}\Big)\ln^2\frac{1+\delta}{|1-\delta|}$$
$$+\delta(3-\delta^2)\Big[\varphi\Big(\frac{1-\delta}{1+\delta}\Big)+2\varphi\Big(-\frac{1-\delta}{1+\delta}\Big)+\frac{\Pi^2}{4}-\frac{3}{4}\ln^2\frac{1+\delta}{|1-\delta|}+\frac{1}{2}\ln\frac{1+\delta}{|1-\delta|}\ln\frac{64\delta^2}{|1-\delta^2|^3}\Big]$$
$$+(3+2\delta^2-\delta^4)[F(\delta^2)+\tfrac{3}{2}G(\delta^2)-H(\delta^2)]\Big\},$$

where

$$\delta=(1+4m^2/k^2)^{1/2},\quad \varphi(x)=\int_1^x\frac{dt}{t}\ln|1+t|,$$

$$F(x)=\int_{-1}^1\frac{dt}{t}\ln(1+t)\ln\Big|1-\frac{t^2}{x}\Big|,$$

$$G(x)=\int_{-1}^1\frac{dt}{1+t}\ln\Big(\frac{1-t}{2}\Big)\ln\Big|1-\frac{t^2}{x}\Big|,$$

$$H(x)=\int_{-1}^1\frac{dt}{1+t}\ln|t|\ln\Big|1-\frac{t^2}{x}\Big|.$$

We note in passing that (as indicated above) the quantity calculated by Källén and Sabry is $\Pi/(1+\Pi)$ rather than Π itself, and one consequently has to subtract the improper self-energy diagrams included by these authors if one wishes to isolate the fourth-order contribution to $\Pi(k^2)$. One readily obtains from the above the Jost-Luttinger result for b_1 as well as the following expression for b_0:

$$b_0=-\{\tfrac{5}{6}+(16/3)[F(1)+\tfrac{3}{2}G(1)-H(1)]\}.$$

Using results of Sandham[9] one finds

$$F(1)=-7/4\zeta(3),$$

$$G(1)=3\zeta(3)-\tfrac{1}{3}\pi^2\ln 2,$$

$$H(1)=\tfrac{7}{2}\zeta(3)-\tfrac{1}{2}\pi^2\ln 2,$$

and the complete expression for b_0:

$$b_0=-\tfrac{5}{6}+4\zeta(3),$$

where $\zeta(3)$ is the Riemann zeta function

$$\zeta(3)=\sum_{n=1}^{\infty}n^{-3}.$$

The insertion of this result into (3) shows that $\varphi(\alpha_0)$ has the form

$$\varphi(\alpha_0)=\tfrac{2}{3}\alpha_0+\alpha_0^2+\alpha_0^3[-(29/36)+(8/3)\zeta(3)].$$

The sign of the α_0^3 term is thus clearly positive and one concludes that to sixth order there is no possibility of satisfying the eigenvalue condition

$$\varphi(\alpha_0)=0$$

essential for the internal consistency of quantum electrodynamics.

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