EFFECTIVE, CENTRAL, DENSITY-DEPENDENT POTENTIAL FROM TENSOR FORCE IN NUCLEAR MATTER

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A local, central, density-dependent potential is derived from a given two-nucleon tensor force in nuclear matter. Such an effective force can be used in finite-nuclei calculations in place of the tensor force to determine the gross properties of nuclei.

The role of the tensor force in nuclear matter is even qualitatively different from that of a central force, so far as binding is concerned. Its contribution to energy does not increase as fast with increasing density as that for a central force, as has been shown by the detailed calculation of Sprung et al.[1] When doing a Hartree-Fock (HF) or Thomas-Fermi (TF) calculation for finite nuclei, generally an effective central interaction is used, but it has density dependence in it coming from two sources: (a) the short-range repulsive part of the potential and (b) the long-range tensor potential. In this note, we shall concentrate on (b) entirely, and give a definite prescription for constructing a local, central, density-dependent potential from the long-range part of the tensor force. Kuo and Brown[2] have pointed out that the second-order Born contribution of the tensor force in nuclei may be approximated by an effective central force in first order, but they did not consider any density dependence in it. Brandow[3] suggested that a central, density-dependent potential could be used in place of tensor force, while Bethe[4] proposed empirically an explicit form of this density dependence in the physically interesting region $k_F = 0.9$ F^{-1} to 1.5 F^{-1}. Bethe showed, by numerical calculation, that the first-order contribution of this effective force to energy is approximately the same as the exact second-order contribution of the tensor potential. However, his approach is essentially empirical, and no definite prescription has till now been given as to how to construct this density-dependent force in a simple manner. Manning and Volkov[5] have demonstrated, by detailed HF calculations, that the density dependence in the force can have important effects in the equilibrium properties of finite nuclei, while Bethe[4] has emphasized its importance to get the gross properties of nuclei from TF calculation. In view of this, it is important to investigate how to construct such a density-dependent central force, given any well-behaved tensor potential. In this note, we shall do so, such that the first-order direct contribution of this central force in nuclear matter will be identical to the second-order direct tensor contribution, and the exchange contributions of the two will also be approximately equal. Applying the local-density approximation, such a force can then be taken over to a finite nucleus to perform HF or TF calculations. As a specific example, the tensor part of the one-pion-exchange potential (OPEP) will be taken, and the effective central force calculated for it. This will be compared with the form that has been suggested by Bethe.[4]

We shall first ignore the spin-isospin summations and include these at the end. Consider first a nondiagonal matrix element of a purely central, local potential v_c:

$$\langle \vec{k}_1, \vec{k}_2 | v_c(r) | \vec{k}_1 + \vec{q}, \vec{k}_2 - \vec{q} \rangle = \frac{1}{\Omega} \int v_c(r) e^{i\vec{q}\cdot\vec{r}} d^3r = F(q). \quad (1)$$

Here Ω is the normalization volume. $F(q)$ is only a function of the magnitude of the momentum transfer q. This is a characteristic of a central, local potential. Nondiagonal matrix elements of the second-order tensor term would be considered to find out under what approximations these can be reduced to the form (1).

Defining the tensor operator $S_{12}(\vec{r}) = 3(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{r})/r^2 - (\vec{\sigma}_1\cdot\vec{\sigma}_2)$, the first-order matrix element of $v_T = v(r)S_{12}(\vec{r})$ is given by

$$I_1 = \langle \vec{k}_1, \vec{k}_2 | v(r) S_{12}(\vec{r}) | \vec{k}_1 + \vec{q}, \vec{k}_2 - \vec{q} \rangle = -(4\pi/\Omega) v_2(q) S_{12}(\vec{q}) \quad (2)$$

with

$$v_2(q) = \int_0^\infty j_2(qr) v(r) r^2 dr.$$

Throughout this paper, j_l stands for the spherical Bessel function of order l. In first-order Born approximation, v_T does not contribute to the binding in nuclear matter, since it vanishes upon angle averaging.

Now consider a second-order nondiagonal matrix element

$$I_2 = \langle \vec{k}_1, \vec{k}_2 | v_T (Q/e) v_T | \vec{k}_1 + \vec{q}, \vec{k}_2 - \vec{q} \rangle,$$

where e is the Rayleigh-Schrödinger propagator, and Q is the Pauli operator ensuring that in the intermediate states, only states above the Fermi sphere come in. Here $\vec{k}_1, \vec{k}_2$ are understood to be within the Fermi sphere. Then

$$I_2 = -\left(\frac{m}{\hbar^2}\right) \sum_{q'} \frac{\langle \vec{k}_1, \vec{k}_2 | v_T | \vec{k}_1 + \vec{q}', \vec{k}_2 - \vec{q}' \rangle \langle \vec{k}_1 + \vec{q}', \vec{k}_2 - \vec{q}' | v_T | \vec{k}_1 + \vec{q}, \vec{k}_2 - \vec{q} \rangle}{\vec{q}' \cdot (\vec{k}_1 - \vec{k}_2 + \vec{q}')}$$

such that $|\vec{k}_1 + \vec{q}'| > k_F$ and $|\vec{k}_2 - \vec{q}'| > k_F$. In the above expression, the summation $\sum_{q'}$ may be replaced by $\Omega/(2\pi)^3 \int d^3q'$, and the matrix elements entering the numerator involve expressions given in Eq. (2). It is clear, however, that I_2 depends on the initial momenta $\vec{k}_1, \vec{k}_2$ of the two particles, and not on just the vector $\vec{q}$, as in Eq. (2). Our approximation involves in averaging $\vec{k}_1, \vec{k}_2$ over the Fermi sphere, so that $\bar{I}_2$ may also take the same form as Eq. (1). It can be shown that

$$\int_{\substack{k_1, k_2 < k_F \\ |\vec{k}_1 + \vec{q}'|, |\vec{k}_2 - \vec{q}'| > k_F}} \frac{d^3k_1 d^3k_2}{\vec{q}' \cdot (\vec{k}_1 - \vec{k}_2 + \vec{q}')} \left[\int d^3k_1 d^3k_2 \right]^{-1} = \frac{3}{20k_F} \frac{1}{q'} P\left(\frac{q'}{2k_F}\right),$$

where $P(q'/2k_F)$ is a function of the scalar $(q'/2k_F)$ only, and has been defined by Euler[6] and Levinger et al.[7] With the above approximation, it can be simply shown that $\bar{I}_2$ can be written in the same form as in Eq. (1):

$$\bar{I}_2 = \frac{1}{\Omega} \int e^{i\vec{q} \cdot \vec{r}} v_{\text{eff}}(r, k_F) d^3r \tag{3}$$

with

$$v_{\text{eff}}(r, k_F) = -\left(\frac{m}{\hbar^2}\right) \left(\frac{3}{10\pi}\right) v(r) S_{12}{}^2(\hat{r}) \frac{1}{k_F} \int_0^\infty P\left(\frac{q'}{2k_F}\right) v_2(q') j_2(q'r) q' dq', \tag{4}$$

where $v_2(q')$ has been defined in Eq. (2). Thus, starting with a tensor potential $v(r)S_{12}(\hat{r})$, its second-order contribution gives rise to an effective potential

$$v_{\text{eff}}(r, k_F) = -v(r) f(r, k_F) S_{12}{}^2(\hat{r}), \tag{5}$$

where

$$f(r, k_F) = \left(\frac{m}{\hbar^2}\right) \frac{3}{10\pi} \frac{1}{k_F} \int_0^\infty P\left(\frac{q'}{2k_F}\right) v_2(q') j_2(q'r) q' dq'$$

is a dimensionless quantity that modifies the radial form of the effective potential form $v(r)$ and also introduces a density dependence.

Including the spin-isospin sums properly, it will be easy to show that the first-order direct contribution to potential energy of $v_{\text{eff}}(r, k_F)$ is the same as the second-order direct contribution of the tensor potential. To illustrate this, let us take the tensor potential from OPEP:

$$v_T = a(\vec{\tau}_1 \cdot \vec{\tau}_2) \left(1 + \frac{3}{\mu r} + \frac{3}{\mu^2 r^2}\right) \frac{e^{-\mu r}}{r} S_{12}(\hat{r}) = (\vec{\tau}_1 \cdot \vec{\tau}_2) v(r) S_{12}(\hat{r}), \tag{6}$$

where μ is the one-pion range and $a = \frac{1}{3} g^2 (m_\pi/m)^2$. Then the effective potential, with our prescription,

is

$$v_{\text{eff}}(r,k_F) = -v(r)f(r,k_F)(\vec{\tau}_1\cdot\vec{\tau}_2)^2 S_{12}^2(\hat{r}). \tag{7}$$

The second-order direct contribution to potential energy per particle is given by

$$\frac{E_d^{(2)}}{A} = -\frac{2m}{\hbar^2}\frac{1}{20\pi^2}k_F^4\int_0^\infty P(u)[v_2(2uk_F)]^2 u\,du \times 12\times 24, \tag{8}$$

the factor 12×24 coming from spin-isospin sum, which is equivalent to $\mathrm{Tr}[\vec{\tau}_1\cdot\tau_2]^2\,\mathrm{Tr}\langle S_{12}^2\rangle$. Here, by E_d,[2] we denote the second-order direct term. All the other symbols have been explained before. The expression (8) checks with Eq. (6) of Dahlblom et al.[8]

If we now take the effective potential (7), and calculate the first-order direct term, it is given by

$$\frac{\mathcal{E}_d^{(1)}}{A} = -\frac{1}{12\pi}k_F^3\int_0^\infty v(r)f(r,k_F)r^2 dr\times 12\times 24. \tag{9}$$

Substituting for $f(r,k_F)$ and simplifying, we get a result identical to the right-hand side of (8). The first-order exchange contribution to potential energy of $v_{\text{eff}}(r,k_F)$ is given by

$$\frac{\mathcal{E}_{\text{ex}}^{(1)}}{A} = \frac{1}{2\pi^2}\int_0^{k_F} j_0(2kr)\left(1-\frac{3}{2}\frac{k}{k_F}+\frac{1}{2}\frac{k^3}{k_F^3}\right)k^2 dk\, v(r)f(r,k_F)d^3r\times 24\times -6, \tag{10}$$

where $\vec{k}=\frac{1}{2}(\vec{k}_1-\vec{k}_2)$. To estimate this term roughly, we shall take only the long-range part of v_T in (6), i.e., assume that $v_T(r)=0$ for $r<r_0$ and is given by (6) for $r\geq r_0$. Typically, $r_0\sim 1$ F in the Moszkowski-Scott[9] spirit. For such values of r, it is a good approximation to write

$$\int_0^{k_F} j_0(2kr)\left(1-\frac{3}{2}\frac{k}{k_F}+\frac{1}{2}\frac{k^3}{k_F^3}\right)k^2 dk \approx \frac{1}{24}k_f^3 j_0(k_F r). \tag{11}$$

So we get

$$\frac{\mathcal{E}_{\text{ex}}^{(1)}}{A} \approx -\frac{1}{12\pi}k_F^3\int j_0(k_F r)v(r)f(r,k_F)r^2 dr\times 12\times 12.$$

Since for the physically interesting region of k_F and r_0 between 1 and 1.6 F we are well within the first zero of j_0 (i.e., $k_F r_0<\pi$), we can safely write

$$\frac{\mathcal{E}_{\text{ex}}^{(1)}}{A} < -\frac{1}{12\pi}k_F^3 j_0(k_F r_0)\int v(r)f(r,k_F)r^2 dr\times 12\times 12.$$

Comparison with the direct term (9) gives

$$\mathcal{E}_{\text{ex}}^{(1)}/A < \tfrac{1}{2}j_0(k_F r_0)\,\mathcal{E}_d^{(1)}/A.$$

Thus, for $k_F=1.4$ F^{-1}, $r_0=1$ F, $\mathcal{E}_{\text{ex}}^{(1)}/\mathcal{E}_d^{(1)}<35\%$; while if $r_0=1.6$ F, $\mathcal{E}_{\text{ex}}^{(1)}/\mathcal{E}_d^{(1)}<18\%$. The corresponding values, calculated from exact second-order calculations[8] (for $k_F=1.3$ F^{-1}) are $E_{\text{ex}}^{(2)}/E_d^{(2)}=22\%$ for $r_0=1$ F and $E_{\text{ex}}^{(2)}/E_d^{(2)}=10\%$ for $r_0=1.6$ F.

The quantity $v_2(q)$, as defined in Eq. (2), is expressible analytically[8] for a $v(r)$ of the form defined in (6), with an arbitrary cutoff r_0. In Fig. 1, we have plotted $f(r,k_F)$ versus r for $r_0=1$ F and for various values of k_F [a certain constant multiplicative factor of $f(r,k_F)$ has been omitted]. It will be seen that this function has a fairly strong r dependence, so that $v_{\text{eff}}(r,k_F)$ cuts off more sharply than $v(r)$. This r dependence is roughly the same for the values of k_F from 0.9 to 1.5 F^{-1}, at least in the important region of r from 1 to 2 F, beyond which $v_{\text{eff}}(r,k_F)$ is not appreciable. Further, in this region the effective potential is falling off almost linearly with k_F, which is one of the forms which Bethe[4] also suggested. These characteristics persisted when the cutoff r_0 was changed to 1.4 F. However, because of the strong r dependence of $f(r,k_F)$, we cannot write the ratio of triplet to singlet po-

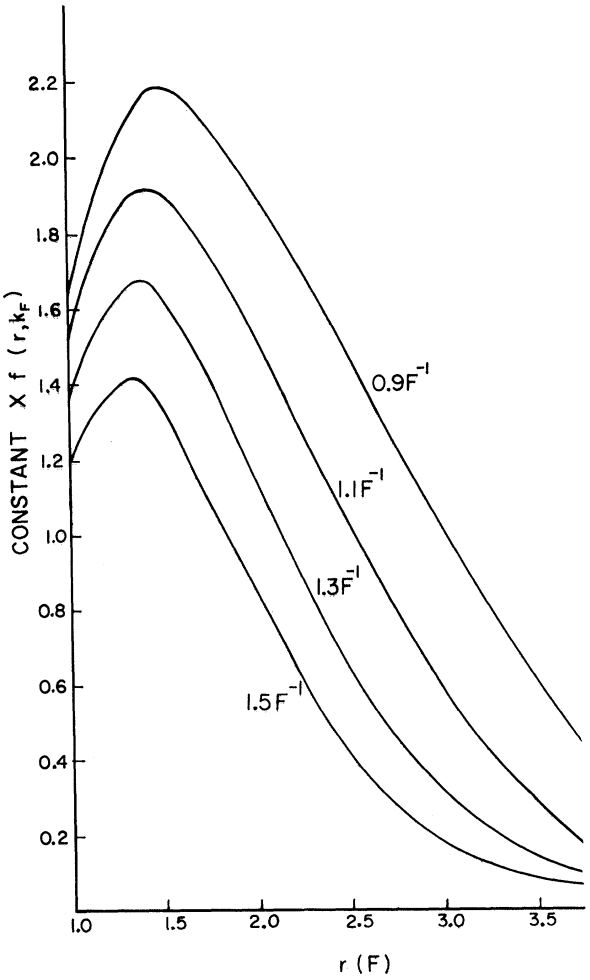

FIG. 1. Variation of $f(r, k_F) \times$ const defined in Eq. (5) with r for various values of k_F. The values of k_F are indicated on each curve in units of F^{-1}.

tential in the simple form that Bethe[4] expressed. We are of the opinion that since $f(r, k_F)$ involves only one numerical integration for a given form of tensor force, it should be calculated explicitly while doing a HF calculation. Finally, it may be mentioned that third-order terms in which the same two particles are interacting twice via the tensor force and once via the central force can similarly be expressed by some effective density-dependent central potential, which would add to the potential (5). It will have a form like $v(r)g(r, k_F)S_{12}{}^2(\vec{r})$, where $g(r, k_F)$ is a function which would modify the r dependence of $v(r)$ and introduce a density dependence.

To sum up, we have shown how the role of the tensor force in nuclear matter can be simulated by a local, central, density-dependent potential. Such an effective potential should prove useful in HF calculations of spherical finite nuclei.

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