Table I. Data points used in determining best fit.

Decay	Branching ratio	Reference
$\Lambda \to p + e^- + \bar{\nu}$	$(8.8 \pm 1.5) \times 10^{-4}$	a
$\Sigma^- \to n + e^- + \bar{\nu}$	$(1.25 \pm 0.17) \times 10^{-3}$	a
$\Sigma^- \to \Lambda + e^- + \bar{\nu}$	$(6.4 \pm 1.2) \times 10^{-5}$	b
$\Xi^- \to \Lambda + e^- + \bar{\nu}$	$(1.0_{-0.65}^{+1.3}) \times 10^{-3}$	c
Decay	g_A/g_V	**Reference**
$n \to p + e^- + \bar{\nu}$	-1.25 ± 0.04	d
$\Lambda \to p + e^- + \bar{\nu}$	$-1.14_{-0.33}^{+0.23}$	e
$\Sigma^- \to n + l^- + \bar{\nu}$	$0.05_{-0.32}^{+0.23}$	This work

[a]A. H. Rosenfeld, N. Barash-Schmidt, A. Barbaro-Galtieri, L. R. Price, P. Söding, C. G. Wohl, M. Roos, and W. J. Willis, Rev. Mod. Phys. 40, 77 (1968). The authors compile data from several experiments.

[b]Barash et al., Ref. 9.

[c]J. R. Hubbard, J. P. Berge, and P. M. Dauber, Phys. Rev. Letters 20, 465 (1968).

[d]G. Conforto, Acta Phys. Acad. Sci. Hung. 22, 15 (1967). This value of g_A/g_V is obtained from experiments involving free-neutron correlations. If one includes the determination from nuclear physics, a best value of -1.20 ± 0.02 is obtained. It is felt, however, that the free-neutron data are more reliable.

[e]G. Conforto, in Proceedings of the International School at Herceg-Novi, Yugoslavia, 1965, edited by M. Nikolić (Secretariat du Department de Physique Corpusculaire, Centre de Recherches Nucléaires, Strassbourg-Cronenbourg, France, 1965). This value is compiled from four experiments: V. G. Lind et al., Phys. Rev. 135, B1483 (1964); C. Baglin et al., Nuovo Cimento 35, 977 (1965); R. P. Ely et al., Phys. Rev. 137, B1302 (1965); J. Barlow et al., Phys. Letters 18, 64 (1965).

The χ^2 is 4.45, and the confidence level is 35%.

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[1]N. Cabibbo, Phys. Rev. Letters 10, 531 (1963).

[2]W. Willis et al., Phys. Rev. Letters 13, 291 (1964).

[3]R. Bangerter, A. Barbaro-Galtieri, J. P. Berge, J. J. Murray, F. T. Solmitz, M. L. Stevenson, and R. D. Tripp, Phys. Rev. Letters 17, 495 (1966); M. Watson, M. Ferro-Luzzi, and R. D. Tripp, Phys. Rev. 131, 2248 (1963).

[4]Events for which the Σ^- length is less than 1 mm are excluded to insure that all the events are indeed Σ^- decays.

[5]G. Källén, Elementary Particle Physics (Addison-Wesley Publishing Company, Inc., Reading, Mass., 1964), p. 361.

[6]We adopt the following notation: The Hamiltonian for leptonic baryon decay is $H = (G/\sqrt{2}) J_\lambda l_\lambda$, where l_λ is the usual lepton current, $\bar{u}_e \gamma_\lambda (1+\gamma_5) u_\nu$, and $J_\lambda = V_\lambda + A_\lambda$ is the baryon current. With $V_\lambda = g_V \gamma_\lambda$ and $A_\lambda = -g_A \gamma_\lambda \gamma_5$, the baryon matrix element is then expressed as $\langle n | \gamma_\lambda (g_V - g_A \gamma_5) | \Sigma^- \rangle$. If time-reversal invariance is assumed, g_V and g_A are both real. Equation (3) neglects recoil effects and some terms involving the lepton mass. D. R. Harrington, Phys. Rev. 120, 1482 (1960), gives more exact expressions, but does not include energy dependence of the form factors.

[7]From the known branching ratio of $\Sigma^- \to \Lambda + e^- + \bar{\nu}$ there should be 1.4 events of that type in which the Λ decays neutrally. This contamination cannot seriously affect our results.

[8]To clarify conventions the corresponding expressions for neutron beta decay are $g_A = (D_A + F_A)\cos\theta$ and $g_V = (D_V + F_V)\cos\theta$, yielding $(g_A/g_V)_{n \to p} = D_A + F_A$.

[9]N. Barash, T. B. Day, R. G. Glasser, B. Kehoe, R. Knop, B. Sechi-Zorn, and G. A. Snow, Phys. Rev. Letters 19, 181 (1967).

[10]N. Brene, L. Veje, M. Roos, and C. Cronström, Phys. Rev. 149, 1288 (1966).

[11]C. Carlson, Phys. Rev. 152, 1433 (1966).

DETERMINATION OF COUPLING CONSTANTS FROM POLES IN SCATTERING CROSS SECTIONS*

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An optimally convergent polynomial expansion has been used in determining the residues of pion and $\Lambda + \Sigma$ poles from n-p and K^+-p differential scattering cross sections. A new method of conservatively estimating the uncertainty is also described.

The partial-wave expansion of a scattering amplitude $A(p, x = \cos\theta)$ only converges within an ellipse, and therefore does not exploit the full analyticity properties of A. Assuming A is analytic for x in a cut plane C, let us find a function $z(x)$ such that a polynomial expansion in z which is based on the physical region $-1 \leq x \leq 1$ will converge to A throughout C. According to the theory of polynomial approximation,[1] it is necessary and sufficient that $z(x)$ map the physical region and the cuts into a pair of equipotentials for an electrostatics problem in which opposite charges are placed on the image of the physical region and at $z=\infty$. Such a mapping may be obtained in two steps. First the cuts $-\infty$ to $-x_-$ and x_+ to $+\infty$ in the x plane are symmetrized by the transformation

$$w=(x-x_0)/(1-xx_0), \tag{1}$$

where

$$x_0=(x_+-x_-)\times[x_+x_--1+(x_+^2x_-^2-x_+^2-x_-^2+1)^{1/2}]^{-1}.$$

These cuts are then mapped onto an ellipse by[2]

$$z=\sin[\tfrac{1}{2}\pi F(\sin^{-1}w,k)/K(k)], \tag{2}$$

where $k=1/w(x_+)$, and $K(k)$ and $F(\varphi,k)$ are the complete and incomplete elliptic integrals of the first kind.[3] Under this mapping, the image of the physical region $-1\leq x\leq 1$ is $-1\leq z\leq 1$, and the foci of the ellipse are at $z=\pm 1$. The semimajor axis is

$$a=\cosh[\tfrac{1}{2}\pi K((1-k^2)^{1/2})/K(k)]. \tag{3}$$

If A has any poles, they will lie on the real axis inside our ellipse, and can be removed explicitly. After the poles are eliminated, a polynomial expansion will converge in the physical region as the series $\sum R^{-n}$, where[1]

$$R=a+(a^2-1)^{1/2} \tag{4}$$

[i.e., the error after n terms is bounded by $M/(R-\epsilon)^n$ for any $\epsilon>0$]. The partial-wave expansion generally converges much more slowly.[4] If a pole is not eliminated, there will be a slower convergence, described by a number $R_p<R$.[5] Note that one can meaningfully treat explicitly even baryon-exchange poles in meson-baryon scattering amplitudes, which usually lie outside the region where the partial-wave expansion could be sensitive to them.

In order to demonstrate the utility of this transformation in practical data analysis, we have used it to extract coupling constants from fixed-energy differential-cross-section data. (Differential cross sections and amplitudes have the same analyticity properties in x.) In selecting reactions for study, we note that a direct-channel resonance of high spin would give an angular distribution not easily represented by a few terms of our expansion. Furthermore, the residue of the pole should be fairly large, above the error level of the experiment. We therefore examined n-p and K^+-p scattering.

In the Moravscik[6] method, one expands about the pole x_p:

$$(x-x_p)^2\frac{d\sigma}{d\Omega}\bigg|_{\text{expt}}=\sum_n a_n(x-x_p)^n \tag{5}$$

and relates a_0 to the coupling constant. We suggest that a much better method is to require the absence of the poles at $x=x_i$ in an expression of the form

$$T(x)=\prod_i(x-x_i)\left[\left(\frac{d\sigma}{d\Omega}\right)_{\text{expt}}-\left(\frac{d\sigma}{d\Omega}\right)_{\text{Born}}\right]=\sum_n c_n p_n(x), \tag{6}$$

where the p_n are some orthogonal polynomials. This is similar to the method of Ashmore et al.,[7] but the expansion can be obtained by the conventional least-squares techniques, in which the p_n are orthonormalized according to the statistical weights of the data. When we work with the transformed variables z, we use the expression

$$T(z)=\prod_i(z-z_i)\left[\left(\frac{d\sigma}{d\Omega}\right)_{\text{expt}}-\left(\frac{d\sigma}{d\Omega}\right)_{\text{Born}}\right]=\sum_n c_n p_n(z). \tag{7}$$

Multiplication by zeros of the first order assures that a lesser polynomial is needed for a good fit. Electromagnetic contributions to the Born amplitude can be introduced explicitly. A form factor can also be introduced which, of course, does not alter the residue, but reduces the usually high Born cross sections to reasonable values. A theoretical reason for introducing form factors is that our transformed expansion is somewhat sensitive to the behavior of $A(p,x)$ at $x\to\infty$.

In order to use a form factor without introducing additional parameters, we assume that vertex functions can be written in a universal form $\Gamma(t-t_0)/\Gamma(t_p-t_0)$, where t is the square of the four momentum, t_p is the pole position, and t_0 is related to the normal threshold energy We use $\Gamma=1/(t_0-t)$ and assume that $t_0=(E_{\text{th}}+0.7\ \text{BeV})^2$, where E_{th} is the normal threshold energy for the quantum numbers of the pole. The parameter was obtained by fitting to the proton's magnetic form factor. It will be noted that our

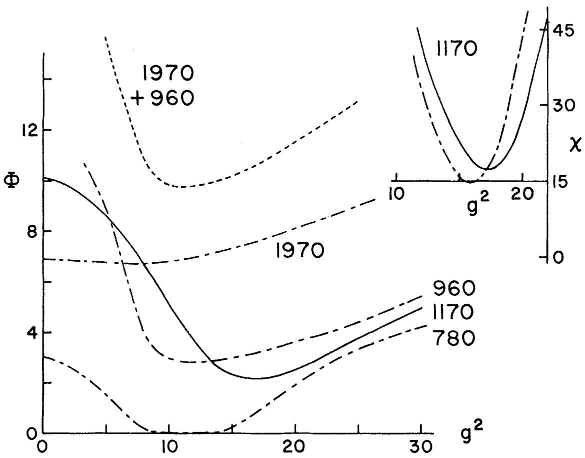

FIG. 1. The inset shows $\chi^2(g^2)$ for the 1170-MeV/c data, with and without a form factor. The main figure shows $\Phi(g^2)$. The notation is defined in Fig. 2.

Ansatz implies that the Born approximation for scattering cross section varies with t somewhat like the fourth power of the electromagnetic form factors.

The inset in Fig. 1 shows how χ^2 varies with g^2 for the 1170-MeV/c K^+-p data when three terms are included in the expansion. On applying the usual criterion,[6,7] we would infer (from the curve obtained when the form factor is omitted) that $g_{\Lambda\Sigma}{}^2 = 15.9 \pm 0.8$; we believe that this criterion systematically underestimates the uncertainty of the determination of g^2 and propose here a new method of estimating the uncertainty which is much more conservative and, we believe, more realistic. The idea is that, instead of determining the uncertainty by the values where $\Delta\chi^2 = 1$, we estimate the maximum allowed $\Delta\chi^2$ by including an estimate of the contribution of the remaining terms of the expansion, assuming that the c_n decrease roughly as R^{-n}.

In order to have a simple way to combine these systematic errors with the statistical errors, we assume that the coefficients c_n are Gaussian random variables with mean zero and variance $v_n = v_0 R^{-2n}$.[8] The function

$$\varphi_n = \min \sum_n [c_n{}^2(1+v_n)^{-1} + \ln(1+v_n)], \tag{8}$$

where the minimization is carried out with respect to v_0 and to the v_n for $n > N$ [i.e., $v_n = \max(0, c_n{}^2 - 1)$ for $n > N$], is essentially -2 times the logarithm of the likelihood function. (The c_n are normalized so that their statistical error is ± 1.) We define the convergence test function Φ to be

$$\Phi = \varphi_M - \varphi_n, \tag{9}$$

where M is to be big enough that $\varphi_m - \varphi_M$ $(m > M)$ is independent of g^2, and where N is the first value for which the minimum χ^2 indicates a good fit to the data (N is shown inside parentheses in Table I). However, it is usually necessary to replace N by $N+1$ in the neighborhood of the minimum χ^2, in order to avoid discriminating against values of g^2 for which the remaining terms in the expansion are smaller than expected. This adjustment can be made in such a way that $\Phi(g^2)$ remains a continuous function. The function Φ is a measure of the degree to which the nearby pole, as a result of not being correctly subtracted out, upsets the rate of decrease of the c_n for $n > N$ that would be expected from the size of the ellipse of meromorphy. We have normalized the convergence test function Φ in such a way that it has roughly the same meaning as χ^2: $\Delta\Phi \simeq 1$ is taken as one standard deviation, $\Delta\Phi \simeq 4$

Table I. Pion-nucleon coupling constant from n-p scattering and the sum $g_{\Lambda\Sigma}{}^2 = g_{\Lambda NK}{}^2 + g_{\Sigma NK}{}^2$ from K^+-p scattering. The numbers in parentheses are the number of terms required for a good fit. A question mark means that the limit of error is uncertain, but very large.

Scattering process	Conventional expansion		Transformed expansion	
	Without form factor	With form factor	Without form factor	With form factor
$n+p$, 350 MeV	14.0 ± 1.9 (7)	14.0 ± 1.3 (7)	$14.5^{+1.4}_{-1.0}$ (6)	$14.1^{+1.4}_{-0.8}$ (6)
$n+p$, 400 MeV	$11.5 \pm ?$ (8)	$14.3^{+1.6}_{-1.9}$ (8)	$14.3^{+1.4}_{-1.6}$ (6)	$13.9^{+1.6}_{-1.7}$ (6)
$K^+ + p$, 1170 MeV/c	(Not determined)		$14^{+?}_{-3.5}$ (3)	17^{+9}_{-4} (3)
$K^+ + p$, 960 Mev/c +1970 MeV/c	...		...	11^{+8}_{-3} (3 + 6)
$K^+ + p$, 780 MeV/c	(Not determined)		$8^{+?}_{-4}$ (3)	12 ± 8 (3)

as two standard deviations, etc.[9] In the tables we quote a "best estimate" of g^2 based on the minimum of Φ, and an uncertainty calculated as half the width of a (roughly 95%) fiducial interval; in other words, the uncertainty is taken as half the distance to the point where $\Delta\Phi = 4$.

The results of analysis of n-p scattering data at 350 MeV[7,10] and at 400 MeV[11] are shown in Fig. 2 and Table I. We call attention to the following points: The uncertainty we quote for the conventional method is much higher than that quoted in Ref. 7. The form factor makes a surprisingly dramatic improvement in the conventional method. The coupling constants determined at the two energies are consistent with each other and with the value obtained from the forward n-p dispersion relation, which is[12] 14.4 ± 0.4.

Experimental K^+-p differential cross-section data are now available at a number of energies.[13] We ignore the mass difference of the Λ and Σ, combining the two poles into one[14] which has a residue depending on the sum of the residues $g_{\Lambda NK}^2 + g_{\Sigma NK}^2 = g_{\Lambda\Sigma}^2$. There have been contradictory estimates of $g_{\Lambda\Sigma}^2$ from forward dispersion relations[15] as well as from photoproduction data.[16] It is, therefore, of some interest to have an independent determination from fixed-energy data, in which the model-dependent errors are controllable.

We show in Fig. 1 and in Table I our results, as obtained with the form factor described above (this form factor is actually quite close, numerically, to the ones usually used in the static model). Without a form factor, we obtained essentially the same lower limits for $g_{\Lambda\Sigma}^2$, but no clear upper limits. The conventional expansion gave, of course, no determination at all.[17]

We wish to emphasize that the uncertainties quoted in the tables arise primarily from the systematic errors and not from the statistical uncertainty of the data. It is therefore not proper to combine statistically the determinations made at nearby energies, because the effective potentials are certainly not independent. The right way to combine the data would be through a proper energy-dependent analysis. However, in order to give an indication of the results that might be obtained from such an analysis, we have added Φ(960 MeV/c) to Φ(1970 MeV/c); at these two energies the scattering cross sections have very different appearances, and our expansion required twice as many terms at 1970 as at 960 MeV/c. Besides, the inelastic cross sections

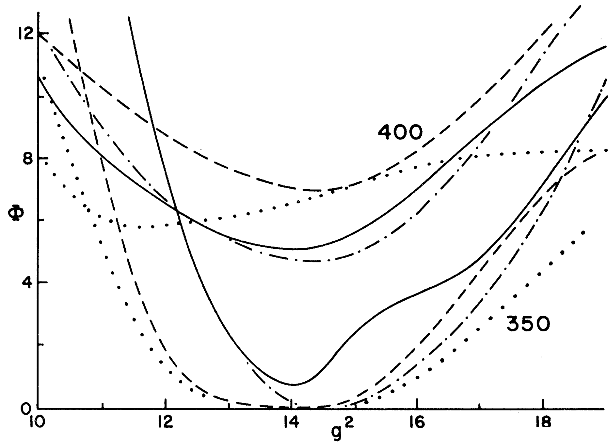

FIG. 2. Plots of the convergence test function $\Phi(g^2)$ for n-p scattering. Notation: dotted line, conventional expansion without form factor; dash, conventional expansion with form factor; dot-dash, new expansion without form factor; solid, new expansion with form factor.

are quite different. If we average the two best upper limits and lower limits given in Table I, we obtain the estimated best value and fiducial half-ranges to be

$$g_{\Lambda\Sigma}^2 = 15_{-4}^{+6}. \tag{10}$$

The estimate from the forward dispersion relations is[15] $g_{\Lambda\Sigma}^2 = 13 \pm 3$.

In conclusion, we emphasize the following points: The full use of analyticity properties is of great practical advantage in analyzing data. To some extent, this has been obscured in the past by overly optimistic estimation of uncertainties. It is important to make use of theoretical knowledge about the rate of convergence of approximations in such an estimation. The fact that good agreement is obtained between coupling constants inferred from fixed-energy and from fixed-momentum-transfer analyses is comforting both for SU(3) and for the analytic properties of the S matrix, even though the uncertainties are at present still large.

We suggest that our transformation may be very useful in a partial-wave analysis. Some further developments which may be of value are treatment of the energy dependence of the expansion coefficients, consideration of the possibility of analytic continuation beyond the branch cuts, and application to production processes.

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[1]J. L. Walsh, Interpolation and Approximation by Rational Functions in the Complex Domain, American Mathematical Society Colloquium Publications, Vol. 20 (American Mathematical Society, New York, 1956), 2nd. ed., Chap. III-VI.

[2]Z. Nehari, Conformal Mapping (McGraw-Hill Book Company, Inc., New York, 1952).

[3]This mapping is easily calculated by use of Gauss transformations.

[4]The idea of increasing the rate of convergence by a conformal mapping has been discussed by W. R. Frazer, Phys. Rev. 123, 2180 (1961). Frazer suggested mapping the cut plane into a circle. In fact, in the applications we consider in this paper, the ellipse is really very nearly circular. However, we also obtain a unique prescription for centering the physical region within the region of convergence.

[5]It will be noted that the mapping $z(x)$ is not determined uniquely by the requirement that a polynomial expansion converge throughout C. However, the numbers R and R_p are invariants.

[6]J. G. Taylor, M. J. Moravcsik, and J. L. Uretsky, Phys. Rev. 113, 689 (1959); P. Cziffra and M. J. Moravcsik, Phys. Rev. 116, 226 (1959).

[7]A. Ashmore et al., Nucl. Phys. 36, 258 (1962).

[8]This implies a certain hypothesis about "random potentials" which we do not wish to put forth as a fundamental proposition; the advantage of the assumption is that it lends itself to a simple machine computation.

[9]Another simple procedure is to plot the $|C_n|$ on logarithmic graph paper and draw the straight line with the theoretical slope that goes through the point for $n \leq N$ for which $(C_n)R^{-n}$ is greatest. Then the estimated contribution to $\Delta\chi^2$ from the remaining terms can be read from the graph. We have checked with this method our error estimates and our estimated rate of convergence. We found from the n-p data, using the conventional expansion in $\cos\theta$, that the rate of convergence seemed to correspond to a singularity at about $(350\ \mathrm{MeV})^2$, rather than $4m_\pi^2 = (270\ \mathrm{MeV})^2$, and used this slightly larger value in subsequent calculations of the error estimate for the conventional expansion. Note that the mass for the effective singularity is much less than the mass of the rho meson.

[10]The data used by Ashmore et al. were augmented by two more points based on interpolation between 300 and 400 MeV, with conservative errors: at 15°, 4.15 ± 2.0; at 25°, 3.22 ± 1.5 (in mb/sr).

[11]A. J. Hartzler and R. T. Siegel, Phys. Rev. 95, 185 (1954); A. J. Hartzler, R. T. Siegel, and W. Opitz, Phys. Rev. 95, 591 (1954).

[12]J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. 35, 737 (1963).

[13]778 MeV/c: S. Focardi et al., Phys. Letters 24B, 314 (1967). 1970 and 1170 MeV/c: W. Hirsch and G. Gidal, Phys. Rev. 135, B191 (1964). 960 MeV/c: preliminary Berkeley data quoted by A. T. Lea, B. R. Martin, and G. C. Oades, Phys. Rev. 165, 1770 (1968).

[14]We have checked for the 1170-MeV/c data that similar results are obtained when the two poles are treated separately.

[15]J. K. Kim, Phys. Rev. Letters 19, 1079 (1967); C. H. Chan and W. L. Yen, Phys. Rev. 165, 1565 (1968); C. H. Chan and F. T. Meiere, Phys. Rev. Letters 20, 568 (1968).

[16]C. W. Peck, Phys. Rev. 135, B830 (1964).

[17]R. L. Warnock and G. Frye, Phys. Rev. 138, B947 (1965). Expressing the branch cuts in terms of known resonances, they estimated $g_\Lambda^2 = -3 \pm 17$ and $g_\Sigma^2 = -20 \pm 15$.