CHARGED-PARTICLE CONTAINMENT IN rf-SUPPLEMENTED MAGNETIC MIRROR MACHINES

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We study the containment of particles in the loss cone of a magnetic-mirror machine by means of rf fields whose frequency slightly exceeds the maximum value of the ion cyclotron frequency within the plasma. We report computations which verify the adiabatic theory of this confinement system.

The rather discouraging outcome of a number of recent thermonuclear reactor design studies based upon unaided magnetic-mirror plasma confinement[1,2] has emphasized the potential value of a device which could prevent or diminish the loss of particles along field lines in such machines, without dissipating an excessive amount of energy in the process. It has long been known[3,4] that radio-frequency fields can in principle be used for this purpose; but their usefulness has been questioned on the grounds that the rf frequency must either coincide with the (ion or electron) cyclotron frequency, in which case there is an irreversible and probably prohibitive dissipation of rf energy in the plasma, or differ substantially from it, in which case the rf fields act by exerting radiation pressure and the required field strengths lead to a prohibitive power loss in the metal structure used to localize them. It has recently been pointed out[5] that a compromise is possible—to use rf fields whose frequency slightly exceeds the maximum value of the ion cyclotron frequency at points accessible to the plasma; and calculations based upon the conservation of certain adiabatic invariants of the particle motion have shown that the thermonuclear prospects of this approach are rather encouraging. In this note we report the outcome of a series of computations designed to test the underlying adiabatic theory.

We selected an axisymmetric parabolic magnetic mirror profile

$$\vec{B} = B_0\left(-\frac{xz}{L_B^2},\ -\frac{yz}{L_B^2},\ 1+\frac{z^2}{L_B^2}\right),$$

supplemented by a circularly polarized standing wave with a node at the center of the well, $\vec{E} = E_0(z/L_E)(\cos\omega t, -\sin\omega t, 0)$ and $B_{rf} = (c/\omega z)\vec{E}$, where L_E and L_B are arbitrary scale lengths. The values of B_0 and ω are assumed to be such that the cyclotron frequency $\Omega = eB/mc$ of an ion of charge e and mass m is lower than ω at the midpoint $x=y=z=0$. In consequence, there exist resonant surfaces intersecting the z axis at $\pm z_{res}$ at which $\omega = \Omega$. According to the adiabatic theory, such a field configuration should confine a plasma in the neighborhood of the midpoint provided that the distribution function is truncated in velocity space in such a way that no particles can reach the resonant surfaces. The nonrelativistic equations of motion in such fields can be reduced to a dimensionless form convenient for computing if time is measured in units of $1/\omega$ and length in units of c/ω:

$$\dot{v}_x = v_y\Omega_0(1+\alpha z^2) + v_z\Omega_0\alpha yz + g_0 z\cos t + v_z g_0 \sin t, \qquad (1)$$

$$\dot{v}_y = -v_x\Omega_0(1+\alpha z^2) - v_z\Omega_0\alpha xz - g_0 z\sin t + v_z g_0\cos t, \qquad (2)$$

$$\dot{v}_z = \Omega_0\alpha z(v_y x - v_x y) - g_0(v_x\sin t + v_y\cos t), \qquad (3)$$

where $\Omega_0 = eB_0/m\omega c$, $g_0 = eE_0/m\omega^2 L_E$, and $\alpha = c^2/\omega^2 L_B^2$. These equations are invariant under the scaling $\vec{v}\to\beta\vec{v}$, $\vec{r}\to\beta\vec{r}$, $\alpha\to\alpha/\beta^2$, and possess one exact invariant, equal to the particle energy in the (rotating) frame in which $\vec{E}$ vanishes. This invariant was used to check the accuracy of the computations; it remained constant to within 0.1% in all the cases discussed here.

The three adiabatic invariants discussed in Ref. 5 here take the form

$$\epsilon_{rf} = \tfrac{1}{2}(v_z - v_{Ez})^2 + \mu_{rf}\Omega + \psi, \qquad (4)$$

$$\mu_{rf} = \tfrac{1}{2}[(v_x - v_{Ex})^2 + (v_y - v_{Ey})^2]/\Omega, \qquad (5)$$

$$J_{rf} = \oint(\epsilon_{rf} - \mu_{rf}\Omega - \psi)^{1/2} ds, \qquad (6)$$

where

$$\Omega=\Omega_0[(1+\alpha z^2)^2+\alpha^2z^2(x^2+y^2)]^{1/2},$$

$$v_{Ex}=g_0z\sin t+[-\Omega_0{}^2\alpha^2z^3xg_0(x\sin t+y\cos t)+\Omega_0(1+\alpha z^2)g_0z\sin t]/(1-\Omega^2),$$

$$v_{Ey}=g_0z\cos t+[-\Omega_0{}^2\alpha^2z^3yg_0(x\sin t+y\cos t)-\Omega_0(1+\alpha z^2)g_0z\cos t]/(1-\Omega^2),$$

$$v_{Ez}=[\Omega_0{}^2(2+\alpha z^2)\alpha z^2g_0(x\sin t+y\cos t]/(1-\Omega^2),$$

$$\psi=\tfrac{1}{2}g_0{}^2z^2[1+\Omega_0(1+\alpha z^2)-\tfrac{1}{2}\Omega_0{}^2\alpha^2(x^2+y^2)z^2]/(1-\Omega^2),$$

and $\oint ds$ indicates an integral along a field line of $\vec{B}$. It is readily shown that $J_{\rm rf}$ is a function of $\chi=r^2(1+\alpha z^2)$, the stream function for $\vec{B}$, and for simplicity we have taken χ instead of $J_{\rm rf}$ as the invariant.

The adiabatic theory can be used to predict the behavior of a particle injected at the point $x=y=z=0$, at $t=0$, with an initial velocity $v_z=v_\parallel$, $v_\perp=(v_x{}^2+v_y{}^2)^{1/2}$, and $\varphi=\tan^{-1}(v_y/v_x)$. Let us consider first the case of a particle injected along the axis of the magnetic field $\vec{B}_0$, (i.e., $v_\perp=0$, $\mu_{\rm rf}=0$). For $g_0=0$, the particle is lost from the system. For a given finite g_0, however, as Eq. (4) shows, such particles will be reflected at a point $z\leq z_{\rm refl}$ provided that $\frac{1}{2}v_\parallel{}^2\leq\psi(z_{\rm refl})$ and will in consequence remain adiabatically confined. (The $\leq$ sign here reflects the presence of the relatively small time-dependent term v_{Ez}.) As $v_\parallel$ is increased, $z_{\rm refl}$ will approach $z_{\rm res}$. It can never exactly reach $z_{\rm res}$ for any finite $v_\parallel$, since by definition $\psi\to\infty$ as $z\to z_{\rm res}$. However, when $z_{\rm refl}$ reaches some value $z_{\rm lim}$, slightly inferior to $z_{\rm res}$, we would expect adiabatic theory to break down; and at higher values of $v_\parallel$ we might expect one or the other of two consequences to follow:

(i) The particle is still reflected at a point inside $z_{\rm res}$, so the invariants remain defined, but their values experience a (possibly irreversible) change while the particle is in the "nonadiabatic" region.

(ii) The particle passes through $z_{\rm res}$, at which point the invariants are no longer even defined. If it gets well past $z_{\rm res}$, it will once more enter a region where the adiabatic theory should apply, but the new values of the invariants are not simply related to their values of the other side of the resonance. Since the changes in $\epsilon_{\rm rf}$ and $\mu_{\rm rf}$ depend upon the phase with which the particles enter the resonance zone, we would expect these quantities to change in a stochastic manner at each subsequent transit through resonance and hence for the particle to be lost after a variable but finite number of transits.

To summarize, two or possibly three categories of particles are predicted: (1) adiabatically confined particles with $z_{\rm refl}$ significantly less than $z_{\rm res}$; (2) nonadiabatic and eventually lost particles with $z_{\rm refl}>z_{\rm lim}$ or even $>z_{\rm res}$; and possibly (3) nonadiabatic but confined particles, in which the variations in the invariants are not sufficiently random to result in particle loss.

The above analysis applied to the case of injection along the axis; however, cases where $v_\perp\neq0$ can be easily considered, since the value of $v_\perp$ determines $\mu_{\rm rf}$ and the equation defining the outer bound on the reflection point $z_{\rm refl}$ obtained from Eqs. (4) and (5) is

$$\tfrac{1}{2}v_\parallel{}^2=\tfrac{1}{2}v_\perp{}^2\left(\frac{\Omega(z_{\rm refl})-\Omega(0)}{\Omega(0)}\right)+\psi(z_{\rm refl}). \qquad (7)$$

Adiabatic theory does not provide internally a criterion which enables one to determine $z_{\rm lim}$, which is the value of $z_{\rm refl}$ at which the theory breaks down. However, $z_{\rm lim}$ is expected to be independent of the angle of injection since a particle which is reflected at $z_{\rm lim}$ approaches this point with $v\approx v_\perp$ and $v_\parallel\approx0$, whatever its injection angle. Consequently, if $z_{\rm lim}$ is determined numerically for $v_\perp=0$, Eq. (7) then defines the theoretical outer edge of the zone in velocity space within which particles are confined adiabatically.

In comparing this adiabatic theory with the numerical results, it is necessary to take into account the fact that Eqs. (4) and (5) for the adiabatic invariants are only correct to lowest order in the expansion parameter (essentially v/c); and in view of practical limitations on computing time, this parameter cannot be made arbitrarily small—it was typically around 0.02 in the cases studied. Consequently, variations at least of the order of the expansion parameter in the computed values of $\epsilon_{\rm rf}$ and $\mu_{\rm rf}$ are expected. In default of "exact" expansions for the invariants, we adopted the criterion that such variations were to be regarded as compatible with conservation of the "exact" invariants provided that they were periodic; that is, provided that after a finite number of particle transits, the sequences of values of the approximate invariants repeated themselves exactly.

To test these predictions and to investigate the

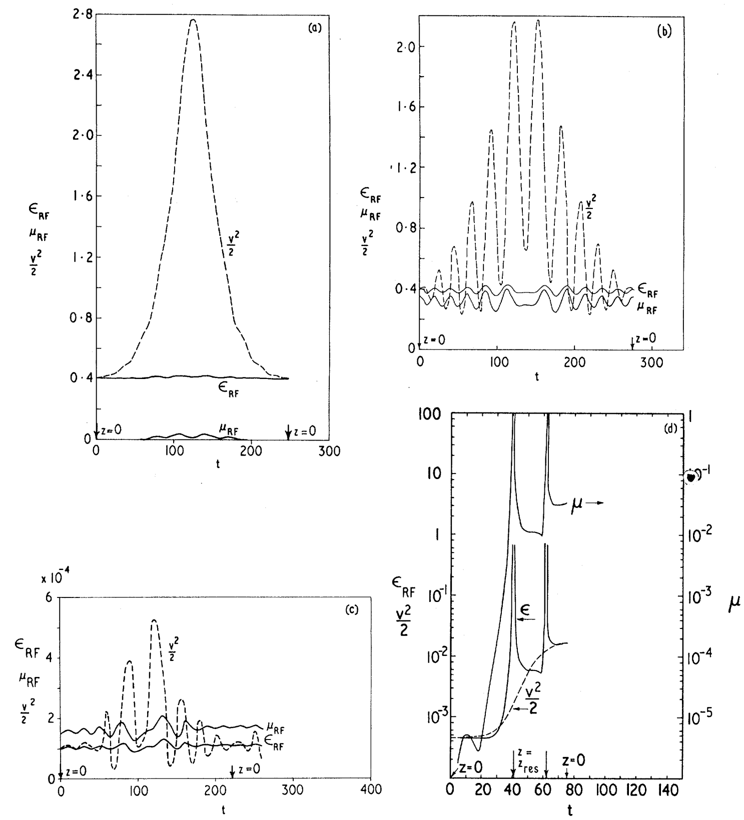

FIG. 1. Time evolution of ϵ_{rf}, μ_{rf}, and energy $v^2/2$ for particles injected at $x=y=z=0, t=0$ for half of a transit period. The injected velocities (v, θ, φ) of the particles are given below: (a) adiabatically confined particle (0.009, 0.0, 0.0); (b) adiabatically confined particle (0.009, 0.57, 0.0); (c) "nonadiabatically confined" particle (0.01414, 0.7954, π); (d) "nonadiabatic lost" particle (0.03, 0.0, 0.0).

accuracy with which these invariants remain constant even when a particle approaches the cyclotron resonance, we integrated Eqs. (1) to (3) numerically, using a differential-equation routine with automatic step length adjustment. Even with given initial conditions, Eqs. (1) to (3) define a three-parameter set of solutions depending on g_0, α, and Ω_0. We report here the most interesting g_0 dependence, taking $\alpha = 0.357$ and $\Omega_0 = 0.675$—values which, with a scaling $\beta = 24.1$, are relevant to an existing experiment,[6] Phoenix II at Culham.

These numerical calculations confirmed the predictions given above in considerable detail. All three categories of particles were discovered. The time dependence of ϵ_{rf} and μ_{rf} for two examples of adiabatically confined particles are shown in Figs. 1(a) and 1(b); and the corresponding trajectories in phase space, projected onto the x-z plane and the $v_\parallel$-$v_\perp$ plane, respectively, are shown in Figs. 2(a) and 2(b). The variations in ϵ_{rf} and μ_{rf} are in both cases small, particularly when compared with the variation in $\frac{1}{2}v^2$; and these variations were exactly periodic, with periods of a few transit times. These figures illustrate rather perspicuously the mode of operation of near-resonance adiabatic fields. Fig. 1(d) rep-

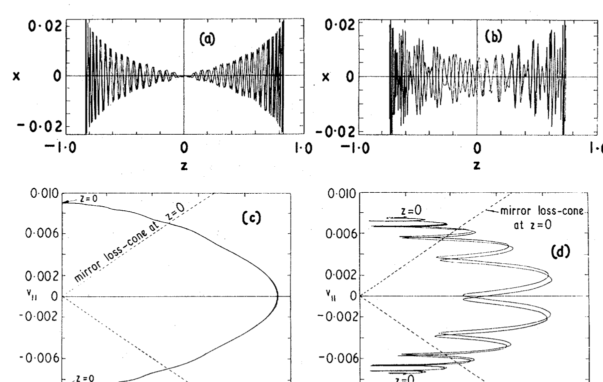

FIG. 2. Trajectories of two adiabatically confined particles for one complete transit period. (a), (c) projections onto the x-z and $v_\parallel$-$v_\perp$ planes, respectively, for the particle with injection parameters (0.009, 0.0, 0.0); (b), (d) for the case (0.009, 0.57, 0.0).

resents a typical category-(2) particle, which passes through the resonance surface at which the values of ϵ_{rf} and μ_{rf} become infinite. It is seen that on both sides of the first passage through resonance, adiabatic behavior is maintained and that a reflection results, leading to a second passage through resonance. At this and each subsequent passage through resonance, a further, apparently stochastic, change in the invariants occurs, and in any real magnetic well such a particle would be lost after a rather small number of transits. For a very narrow range of injection parameters, category-(3) behavior was observed. Fig. 1(c) represents such a particle. It will be seen that the invariants fluctuate by a larger amount than in the case of category-(1) particles; more important, however, is that these fluctuations are nonperiodic but of apparently restricted amplitude. Particles in this class have been followed for 10^5 cyclotron gyration times, during which they remained confined, without any periodicity in ϵ_{rf} and μ_{rf} being detected. Although of theoretical interest, this category represents such a narrow band in injection velocity space that it is of little practical importance. The values of J_{rf} in the above cases are not shown, since for on-axis injection, χ is initially zero and the observed fluctuations in its value do not give a useful measure of its invariance. In cases of off-axis injection, computations showed that its fractional variations were of the same order as those of ϵ_{rf} and μ_{rf}.

In Figs. 3(a) and 3(b), we give in condensed form the results of a very large number of computations designed to map out the boundary between adiabaticity and nonadiabaticity for a fixed g_0 and with varying injection parameters. From the form of Eqs. (4) and (5) for ϵ_{rf} and μ_{rf} it may be seen that, within the zone of adiabaticity, the initial azimuthal phase φ is of no importance. As the edge of this zone is approached, however, the magnitudes and periods of the fluctuations in these invariants were found to depend on phase, and correspondingly the limit of onset of nonadiabaticity is smeared into a phase-dependent band which is too narrow to represent in Fig. 3. The lower solid line in Fig. 3(a) defines the zone of adiabatically confined particles, the shaded band above shows the region of "nonadiabatic" confinement, and finally category-(2) particles lie above this band. These curves may be compared with those in Fig. 3(b) which are contours of constant z_{refl}. The line of adiabatic confinement corresponds rather closely to the contour z_{refl} $=0.83$. If one uses the "empirical" value of z_{lim} $=0.83$ in Eq. (7), one obtains the dot-dashed curve which fits the edge of the adiabatic zone as well as can be expected. The corresponding limiting value of $(1-\Omega)$ is 0.145.

Having confirmed that the adiabaticity boundary could be calculated from Eq. (7), given $v_{\|max}$ $\equiv[2\psi(z_{lim})]^{1/2}$, we investigated the dependence of $v_{\|max}$ on g_0 by considering parallel injection only. For $g_0 \lesssim 0.004$, we found an approximately linear relationship $v_{\|max}=2.1g_0$. For larger values of g_0 a saturation phenomenon is observed, and no further improvement in the confinement is obtained if g_0 is increased beyond 0.013, at which value $v_{\|max}=0.02$.

This general result may be interpreted in the context of two classes of experiment. Firstly, if the scale lengths of electric and magnetic field, L_E and L_B, respectively, are of order c/ω, then deuterons with a parallel energy of 18 MeV could still be reflected <u>if</u> the required electric field strength were available. Secondly, if $L_E \approx L_B$

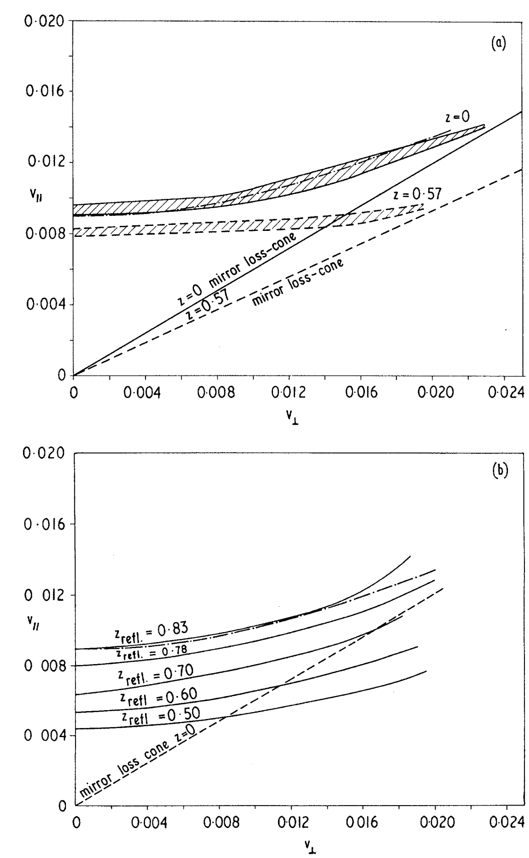

FIG. 3. (a) Boundaries of adiabaticity for injection at $z=0$ and $z=0.57$. (b) Contours of z_{refl} for different injections at $z=0$ in $v_\|-v_\perp$ plane. In both (a) and (b) the dot-dashed line represents Eq. (7).

$\ll c/\omega$, which is the case with most present-day experiments, the results can still be applied, with an appropriate scaling, provided that Ω_0 (essentially the mirror ratio) remains fixed. For example, if we take a set of parameters typical of the Phoenix II experiment[6]: $B(0)=12.5$ kG, $L_E=7$ cm, mirror ratio $R=1+\alpha=1.357$, $\Omega_0=1/1.1R$, the scaling is $\beta=24.1$. The computed result, $v_{\parallel\max}=0.02$ with $g_0=0.0130$, then implies that 0.33-keV protons would be confined by an electric field of 3 kV cm^{-1}. Thus experimentally realizable fields could lead to the trapping of slow ions, which in turn could lead to the build-up of fast ions. Furthermore, since after scaling the unit of v is $L_E\omega$ and that of E is $L_E\omega^2$, for $L_E=14$ cm and $B(0)=16$ kG, protons of 5 keV could be confined by an electric field of 23 kV cm^{-1}.

In extrapolating the results reported here towards a thermonuclear reactor, two questions arise. Firstly, one might ask whether this method of confinement is very sensitive to collisions in the mirror region. A partial answer can be obtained by regarding a collision as equivalent to a reinjection of the particle at $z\neq 0$ with new values of $v_\parallel$ and v_L. To emphasize that a particle can easily remain confined after such an event, we have indicated in Fig. 3(a) the adiabatic boundary for particles injected at $z=0.57$. Secondly, it may be asked how far the above discussion requires modification to take into account the influence of the plasma on the electric field profile which would exist inside it. As was shown in Ref. 5, the self-consistent profile can be calculated if the adiabatic invariants are assumed to be conserved. Such profiles are more complicated than the linear profile adopted here, and the adiabaticity of particles in such profiles is currently under investigation.

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NEW ECHO PHENOMENA IN SUPERCONDUCTORS AND IN NORMAL METALS

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Echo phenomena were observed in normal and superconducting powdered metals. They are believed to be related to magnetoacoustic modes.

In a recent Letter[1] Goldberg, Ehrenfreund, and Weger reported the observation of echoes in superconducting powders. These echoes are somewhat similar to spin echoes[2] and to cyclotron echoes observed in plasmas.[3] The authors suggest this phenomenon may be connected with the motion of pinned fluxoids. We had made similar observations[4] in powdered lead alloys below their critical field H_{c2}. We also observed[4] the same type of echoes in nonsuperconducting pure metals (Pb, Cu, In, Sn) when the particle size of the powders was larger than the skin depth.

The experimental results of Goldberg, Ehrenfreund, and Weger agree with ours in practically all aspects. In particular, the decay times of the echo envelopes are of the same order of magnitude for all the samples. The only discrepancy is connected with the field dependence of the echo amplitude. We found an H^2 law for pure lead ($H\gg H_c$). For superconducting lead alloys the behavior is illustrated in Fig. 1. The echo disappears quite abruptly at the phase transition