The second term in the curly bracket may be neglected. Further, we see that for $\omega_a \simeq \omega_0$ and $\omega_a \simeq 2\omega_0$, there is a sharp increase in $J_1(\omega_0)$ and $J_2(\omega_0)$, respectively. The effect of the former in η_{zz} has been observed in the τ_0 measurements of SZ. The simple theory presented above predicts a similar resonance for $\omega_a \simeq 2\omega_0$.

The author wishes to thank Professor L. H. Nosanow for discussions and for a critical reading of the manuscript. He also wishes to thank Professor Zimmerman for communicating his experimental results prior to publication.

*Work supported by U. S. Atomic Energy Commission.

[1]W. Senghaphan and G. O. Zimmerman, Phys. Rev. Letters 20, 371 (1968).

[2]See, for example, A. Abragam, The Principles of Nuclear Magnetism (Clarendon Press, Oxford, England, 1961), p. 64.

[3]M. G. Richards, J. Hatton, and R. P. Giffard, Phys. Rev. 139, A91 (1965).

[4]R. C. Richardson, E. Hunt, and H. Meyer, Phys. Rev. 138, A1326 (1965).

[5]L. H. Nosanow and C. M. Varma, U. S. Atomic Energy Commission Report No. COO-1569-18.

[6]N. Bloembergen and S. Wang, Phys. Rev. 93, 72 (1954).

[7]R. L. Garwin and A. Landesman, Phys. Rev. 133, A1503 (1964).

[8]The excitations in solid helium can be described in terms of "self-consistent" phonons, and diffusive modes.

[9]E. R. Hunt et al., Phys. Rev. 163, 181 (1967).

[10]M. F. Panczyk et al., Phys. Rev. Letters 19, 1102 (1967).

[11]S. R. Hartmann, Phys. Rev. 133, A17 (1964).

RELATION BETWEEN ENHANCED LONGITUDINAL RESISTIVITY AND ENHANCED TRANSVERSE DIFFUSION*

Jerome Weinstock
Environmental Science Services Administration, Boulder, Colorado
(Received 21 March 1968)

It is a fact[1] that the transverse diffusion coefficient $D_\perp$ in turbulent or nonequilibrium plasmas in uniform magnetic fields exceeds the transverse coefficient predicted for equilibrium plasmas. It theoretically follows that such enhanced transverse diffusion should be accompanied by an enhanced transverse electrical conductivity since both transport processes are the consequence of an enhanced drift of electrons across the magnetic field. Indeed, such a theoretical relation has already been deduced.[2]

Of greater significance, however, is that enhanced diffusion across a magnetic field can be accompanied by an anomalous decrease in electrical conductivity parallel to the magnetic field, and by a related anomalous heating. This has been observed[3,4] experimentally, but has not been satisfactorily explained.[4] A calculation of the parallel conductivity in a turbulent plasma has been made by Yoshikawa.[5] However, the analysis was limited to high frequencies and weak turbulence where it predicts very small deviations in the conductivity from the equilibrium collisional value, and cannot explain the large deviations observed. Indeed, it is too difficult, at present, to obtain satisfactory quantitative explanations of either anomalous transverse diffusion or longitudinal resistivity. Such explanations require detailed considerations of nonlinear processes and boundary conditions. A possibly more fruitful attack, for the present, might be to deduce the relationships between various anomalous transport processes, since such relationships might be insensitive to the details of boundaries and nonlinear processes, and therefore can be quantitatively compared with the experiment.

The purpose of the present communication, then, is to derive a general relationship between enhanced transverse diffusion and enhanced parallel resistivity in turbulent plasmas. This relationship is valid for a remarkably wide set of conditions, and is insensitive to the detailed nature or degree of the turbulence and instability driving the turbulence. It enables one to predict what the enhanced parallel resistivity will be from measurements of enhanced transverse diffusion. The other factor that enters is the characteristic ratio of parallel to transverse spectral wavelength. This ratio is sometimes conveniently determined by the geometry of the plasma.

Physically, the present results imply that turbulence produces a large effective collision frequency so that the scattering of electrons across the magnetic field is enhanced (enhanced transverse diffusion) while the motion of electrons along the magnetic field is retarded (enhanced parallel resistivity).

Our method is based on the Kubo-Green[6] formula which expresses the conductivity in terms of current fluctuations. The conductivity can thus be related to Spitzer's[7] diffusion formula by means of an Ohm's law relating current fluctuations to electric field fluctuations. These considerations are distinct from Yoshikawa's, and the results account for great enhancements in parallel resistivity for weak as well as strong turbulence. It is hoped that these results will encourage investigators to report on measurements of parallel resitivity in addition to transverse diffusion.

(I) Parallel resistivity and transverse diffusion. – To relate enhanced resistivity to enhanced diffusion we begin with the well-established Kubo-Green[6] expression for the frequency-dependent conductivity tensor:

$$\sigma(\omega)=(2nKT)^{-1}\int_{-\infty}^{\infty}dt\,e^{i\omega t}\times\langle\vec{J}(\vec{R},t')\vec{J}(\vec{R},t'+t)\rangle, \tag{1}$$

where $\sigma(\omega)$ is the conductivity tensor at frequency ω, $\vec{J}(\vec{R},t)$ is the (fluctuating) current density in the plasma at position $\vec{R}$ at time t, K is Boltzmann's constant, T is the temperature, n is the free electron density, and the angular brackets denote an ensemble average. In a Cartesian coordinate system with basis vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$, and with a uniform magnetic field $\vec{B}$ directed along $\vec{z}$, the parallel resistivity is given by (1) as

$$\sigma_{zz}(\omega)=(2nKT)^{-1}\int_{-\infty}^{\infty}dt\,e^{i\omega t}\times\langle J_z(\vec{R},t')J_z(\vec{R},t'+t)\rangle. \tag{2}$$

Our goal is to relate $\sigma_{zz}(0)$ to the transverse diffusion coefficient $D_\perp$ in turbulent plasmas by combining (2) with Ohm's law and Spitzer's diffusion formula. Let us then restrict ourselves to the conditions implicit in Spitzer's diffusion formula; that is, to homogeneous stationary turbulence with negligible spatial dispersion of the conductivity or diffusion tensors. The fluctuating electric field and current densities are thus related by the following Ohm's-law expression:

$$\vec{J}^\dagger(\vec{R},\omega)=\sigma(\omega)\cdot\vec{E}^\dagger(\vec{R},\omega), \tag{3}$$

where $\vec{J}^\dagger(\vec{R},\omega)$ and $\vec{E}^\dagger(\vec{R},\omega)$ denote the Fourier transforms of the fluctuating current and electric field densities, and are defined by

$$\vec{J}^\dagger(\vec{R},\omega)\equiv\int_{-\tau}^{\tau}dt\,e^{-i\omega t}\vec{J}(\vec{R},t)\quad(\tau\sim\infty), \tag{4}$$

$$\vec{E}^\dagger(\vec{R},\omega)\equiv\int_{-\tau}^{\tau}dt\,e^{-i\omega t}\vec{E}(\vec{R},t)\quad(\tau\sim\infty), \tag{5}$$

and $\vec{E}(\vec{R},t)$ is the electric field at position $\vec{R}$ at time t.

Furthermore, since the plasma is stationary (2) can be written as

$$\sigma_{zz}(\omega)=(4\tau nKT)^{-1}\int_{-\tau}^{\tau}dt'\int_{-\infty}^{\infty}dt\,e^{i\omega t}\times\langle J_z(\vec{R},t')J_z(\vec{R},t'+t)\rangle$$
$$=(4\tau nKT)^{-1}\langle|J_z(\vec{R},\omega)|^2\rangle\quad(\tau\sim\infty), \tag{6}$$

where we have used the reality of $\vec{J}(\vec{R},t)$.

Substituting the z component of (3) into (6) we have

$$\sigma_{zz}(\omega)=(4\tau nKT)^{-1}\langle|\sigma_{zz}(\omega)E_z(\vec{R},\omega)+\sigma_{zy}(\omega)E_y(\vec{R},\omega)+\sigma_{zx}(\omega)E_x(\vec{R},\omega)|^2\rangle. \tag{7}$$

Physically, however, σ_{zy} and σ_{zx} must both be equal to zero since they are proportional to that current flow along the magnetic field which is due to electric fields perpendicular to the magnetic field. Such current flow is zero since the magnetic field is taken to be uniform throughout, and there is no coupling between perpendicular thermodynamic forces and parallel transport. (This is not to be confused with the Hall effect which leads to current flow along $\vec{x}$ due to electric fields along $\vec{y}$.) Mathematically, it can be proven that σ_{zy} and σ_{zx} are zero for homogeneous, axially symmetric turbulence spectra. Equation (7) thus reduces to

$$\sigma_{zz}(\omega)=(4\tau nKT)^{-1}\sigma_{zz}^{2}(\omega)\langle|E_z^\dagger(\vec{R},\omega)|^2\rangle \tag{8}$$

or

$$R_{zz}(\omega)\equiv\sigma_{zz}^{-1}(\omega)=(4\tau nKT)^{-1}\langle|E_z^\dagger(\vec{R},\omega)|^2\rangle, \tag{9}$$

where $R_{zz}(\omega)$ denotes the longitudinal resistivity of the plasma. Furthermore, since the spectrum is stationary in time, (9) can be written as

$$R_{zz}(\omega) = (2nKT)^{-1}\int_{-\infty}^{\infty} dt\, e^{i\omega t} \times \langle E_z(\vec{R}, t')E_z(\vec{R}, t'+t)\rangle. \quad (10)$$

The dc resistivity $R_{zz}(0)$ is given by the limit of (10) at zero ω:

$$R_{zz}(0) = (nKT)^{-1} \times \int_0^{\infty} dt \langle E_z(\vec{R}, t')E_z(\vec{R}, t'+t)\rangle, \quad (11)$$

where we have once again used the stationarity condition.

The right-hand side of (11) can be related to Spitzer's expression for the transverse diffusion coefficient, for either an electrostatic or transverse mode spectrum of turbulence. First we expand $\vec{E}(\vec{R}, t)$ in terms of its Fourier components $\mathcal{E}(\vec{k}, \omega)$, as follows:

$$\vec{E}(\vec{R}, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \sum_{\vec{k}} \mathcal{E}(\vec{k}, \omega)\, e^{i(\vec{k}\cdot\vec{R} + \omega t)} \quad (12)$$

so that

$$\langle \vec{E}(\vec{R}, t')\vec{E}(\vec{R}, t'+t)\rangle = \int \frac{d\omega}{4\pi\tau} e^{i\omega t} \sum_{\vec{k}} \langle \mathcal{E}(\vec{k}, \omega)\mathcal{E}(-\vec{k}, -\omega)\rangle, \quad (13)$$

where we have used the following conditions of stationarity and homogeneity:

$$\langle \mathcal{E}(\vec{k}, \omega)\mathcal{E}(\vec{k}', \omega')\rangle = \langle \mathcal{E}(\vec{k}, \omega)\mathcal{E}(-\vec{k}, -\omega)\rangle \delta_{\vec{k}\vec{k}'}\delta(\omega+\omega')/2\tau \quad (\tau \sim \infty). \quad (14)$$

Here, $\delta_{kk'}$ and $\delta(\omega+\omega')$, respectively, denote Kronecker and Dirac delta functions, and τ is much larger than all the characteristic times of the plasma [see (4) and (5)]. Let us, henceforth, require that the field spectral density can be separated into electrostatic (longitudinal) and transverse parts so that (13) can be expressed as

$$\langle \vec{E}(\vec{R}, t')\vec{E}(\vec{R}, t'+t)\rangle = \int \frac{d\omega}{4\pi\tau} e^{i\omega t} \sum_{\vec{k}} \left\{ \langle |\mathcal{E}_L(\vec{k}, \omega)|^2\rangle \frac{\vec{k}\vec{k}}{k^2} + \langle |\mathcal{E}_T(\vec{k}, \omega)|^2\rangle \left(\mathsf{I} - \frac{\vec{k}\vec{k}}{k^2}\right)\right\}, \quad (15)$$

where $\mathcal{E}_t$ and $\mathcal{E}_T$, respectively, denote the components of $\mathcal{E}$ parallel and transverse to $\vec{k}$, and I denotes the unit dyadic.

(II) Electrostatic turbulence. – Let us first consider the case in which the turbulent spectrum is entirely electrostatic ($\mathcal{E}_T = 0$). (Transverse turbulence is treated afterwards.) The compoents of (15) then yield

$$\langle E_z(\vec{R}, t')E_z(\vec{R}, t'+t)\rangle = \int \frac{d\omega}{4\pi\tau} e^{i\omega t} \sum_{\vec{k}} \langle |\mathcal{E}_L(\vec{k}, \omega)|^2\rangle \frac{k_z^2}{k^2} = \int \frac{d\omega}{4\pi\tau} e^{i\omega t} \sum_{\vec{k}} \langle |\mathcal{E}_L(\vec{k}, \omega)|^2\rangle \frac{k_\perp^2}{k^2}\frac{k_z^2}{k_\perp^2}$$

$$\equiv \langle\langle k_z^2/k_\perp^2\rangle\rangle_L \langle E_x(\vec{R}, t')E_x(\vec{R}, t'+t) + E_y(\vec{R}, t')E_y(\vec{R}, t'+t)\rangle, \quad (16)$$

where $k_\perp^2 \equiv k_x^2 + k_y^2$, and $\langle\langle k_z^2/k_\perp^2\rangle\rangle_L$ is a mean value of $k_z^2/k_\perp^2$ defined by the following mean value theorem:

$$\left\langle\left\langle \frac{k_z^2}{k_\perp^2}\right\rangle\right\rangle_L \int d\omega\, e^{i\omega t} \sum_{\vec{k}} \langle |\mathcal{E}_L(\vec{k}, \omega)|^2\rangle \frac{k_\perp^2}{k^2} = \int d\omega\, e^{i\omega t} \sum_{\vec{k}} \langle |\mathcal{E}_L(k, \omega)|^2\rangle \frac{k_z^2}{k^2}. \quad (17)$$

Substituting (16) into (11), there results

$$R_{zz}(0) = (nKT)^{-1} \left\langle\left\langle \frac{k_z^2}{k_\perp^2}\right\rangle\right\rangle_L \int_0^{\infty} dt \langle E_x(\vec{R}, t')E_x(\vec{R}, t'+t) + E_y(\vec{R}, t')E_y(\vec{R}, t'+t)\rangle. \quad (18)$$

But the integral on the right-hand side of (18) is Spitzer's[7] expression for an anomalous transverse diffusion coefficient,

$$D_\perp = c^2 B^{-2} \times \int_0^\infty dt \langle E_{x,y}(\vec{R}, t') E_{x,y}(\vec{R}, t'+t)\rangle, \qquad (19)$$

so that (18) becomes the following desired result:

$$R_{zz}(0) = \frac{2B^2}{nKTc^2} \left\langle\!\!\left\langle \frac{k_z^2}{k_I^2} \right\rangle\!\!\right\rangle_L D_\perp \quad (\mathcal{E}_T = 0). \qquad (20)$$

Equation (20) describes the general relation between enhanced parallel resistivity and enhanced transverse diffusion in a turbulent plasma in a magnetic field. It is applicable when the turbulence spectrum is homogeneous, stationary, and electrostatic. It has also been assumed that $D_\perp$ is greater than "classical" and can be represented by (19).[7] Most significant, however, is that (20) is substantially insensitive to the detailed nature or degree of the turbulence and instability driving the turbulence. The nature of the turbulence enters in a gross fashion through the factor $\langle\!\langle k_z^2/k_\perp^2 \rangle\!\rangle_L$. This dependence on $\langle\!\langle k_z^2/k_\perp^2 \rangle\!\rangle_L$ is very reasonable since it implies that the resistivity parallel to $\vec{B}$ will be effected by electrostatic waves only to the extent that these waves have components, k_z, parallel to $\vec{B}$, and correspondingly for $D_\perp$ and $k_\perp$.

As for the evaluation of $\langle\!\langle k_z^2/k_\perp^2 \rangle\!\rangle_L$ we note that if the field spectrum is peaked about some characteristic, $\vec{k}^0$, as is frequently the case, then

$$\langle\!\langle k_z^2/k_\perp^2 \rangle\!\rangle_L \approx (k_z^0/k_\perp^0)^2, \qquad (21)$$

so that (20) becomes

$$R_{zz}(0) \approx \frac{2B^2}{nKTc^2} \left(\frac{k_z^0}{k_\perp^0}\right)^2 D_\perp. \qquad (22)$$

[In some cases it happens that the characteristic wavelengths of the spectrum are determined by the geometry of the plasma device so that (20) and (22) become

$$R_{zz}(0) \approx \frac{B^2}{nKTc^2} \left(\frac{d}{L}\right)^2 D_\perp, \qquad (23)$$

where d and L, respectively, denote the diameter and axial length of the device.]

An important consequence of (20) is that $R_{zz}(0)$ can become very large (compared with its "classical" collisional value) for weak as well as strong turbulence, since $D_\perp$ can be anomalously large for weak as well as strong turbulence. This is in marked contrast to Yoshikawa's results for $R_{zz}(0)$, which is tied quite closely to the "classical" collisional value.

To see just how large $R_{zz}(0)$ can be expected to be for the many experiments in which $D_\perp$ is on the order of the Bohm value

$$D_{\perp\mathrm{B}} \approx (16 m_e \Omega_e)^{-1} KT, \qquad (24)$$

where m_e and $\Omega_e \equiv eB(m_e c)^{-1}$, respectively, denote the mass and cyclotron frequency of electrons, we substitute (24) into (20) to obtain

$$R_{zz}(0) \approx \frac{B}{8enc} \left\langle\!\!\left\langle \frac{k_z^2}{k_\perp^2} \right\rangle\!\!\right\rangle_L. \qquad (25)$$

Comparing (25) with the "classical" resistivity $R_{zz}^{\ c}$ [given by

$$R_{zz}^{\ c} = (e^2 n)^{-1} m_e \nu_e, \qquad (26)$$

where ν_e is the electron collision frequency at thermodynamic equilibrium], we have

$$\frac{R_{zz}(0)}{R_{zz}^{\ c}} \approx \frac{\Omega_e}{8\nu_e} \left\langle\!\!\left\langle \frac{k_z^2}{k_\perp^2} \right\rangle\!\!\right\rangle_L. \qquad (27)$$

Since Ω_e is frequently several orders of magnitude greater than ν_e, it follows that the deviation of $R_{zz}(0)$ from "classical" can be extremely large unless $\langle\!\langle k_z^2/k_\perp^2 \rangle\!\rangle_L$ is correspondingly very small.

A criterion for enhanced longitudinal resistivity in turbulent plasmas can, from (20) and (26), be stated as

$$(2D_\perp/\nu_e \lambda_e^2) \langle\!\langle k_z^2/k_\perp^2 \rangle\!\rangle_L > 1, \qquad (28)$$

where $\lambda_e \equiv \{(\Omega_e^2 m_e)^{-1} KT\}^{1/2}$ is the electron Larmor radius. If $D_\perp$ is approximately that of Bohm, (24), then a rough criterion for enhanced $R_{zz}(0)$ is given by

$$(\Omega_e/8\nu_e) \langle\!\langle k_z^2/k_\perp^2 \rangle\!\rangle_L > 1. \qquad (29)$$

(III) Transverse turbulence. – The results of Sec. II all refer to electrostatic turbulence. For transverse turbulence we use (15) with $\mathcal{E}_L = 0$, and then follow the steps of Sec. II to obtain eventually the following transverse turbulence result:

$$R_{zz}(0)=\frac{2B^2}{nKTc^2}\left\langle\!\!\left\langle\frac{k_\perp^{\,2}}{k_\perp^{\,2}+2k_z^{\,2}}\right\rangle\!\!\right\rangle_T D_\perp \quad (\mathcal{E}_L=0), \tag{30}$$

where $\langle\!\langle k_\perp^{\,2}(k_\perp^{\,2}+2k_z^{\,2})^{-1}\rangle\!\rangle$ is defined by

$$\left\langle\!\!\left\langle\frac{k_\perp^{\,2}}{k_\perp^{\,2}+2k_z^{\,2}}\right\rangle\!\!\right\rangle_T \int d\omega\, e^{i\omega t}\sum_{\vec{k}}\langle|\mathcal{E}_T(\vec{k},\omega)|^2\rangle\left(2-\frac{k_\perp^{\,2}}{k^2}\right)=\int d\omega\, e^{i\omega t}\sum_{\vec{k}}\langle|\mathcal{E}_T(k,\omega)|^2\rangle\left(1-\frac{k_z^{\,2}}{k^2}\right). \tag{31}$$

Comparing (30) with (20) we see that $R_{zz}(0)$ increases with increasing $k_z/k_\perp$ for electrostatic turbulence, and decreases with increasing $k_z/k_\perp$ for transverse turbulence.

(IV) Conclusions.—A general relationship has been determined between anomalous longitudinal resistivity and anomalous transverse diffusion in turbulent plasmas. The relationship takes the form (20) for electrostatic turbulence and (30) for transverse turbulence. An important consequence of this relationship is that large enhancements in longitudinal resistivity can occur for even weakly turbulent plasmas. It has also been shown that, whereas it is presently too difficult to obtain satisfactory quantitative explanations of individual enhanced transport processes, it is a simple matter to obtain a quantitative relationship between two different transport processes. It is therefore our opinion that, at the present time, it would be quite useful to investigate and compare such relationships with the enhanced transport experiments that are day by day becoming more abundant and varied. This would certainly not solve the "anomalous" transport problem, but it would provide a quantitative means of testing some consequences of the prevailing notions about "anomalous" transport.

---

[1]See, for example, Conference on Instabilities and Anomalous Loss Processes in Steady-State Plasmas, edited by W. B. Pardo and H. S. Robertson (University of Miami Press, Coral Gables, Fla., 1966), and references therein.

[2]V. I. Petviashvili et al., Nucl. Fusion 5, 315 (1965); R. R. Ramazashvili and N. L. Tsintsadze, in Proceedings of the Seventh International Conference on Ionization Phenomena in Gases, Belgrade, 1965, edited by B. Perovic and D. Losic (Gradjevinska Knjiga Publishing House, Beograde, Yugoslavia, 1966), p. 69; S. Yoshikawa and D. J. Rose, Phys. Fluids 5, 334 (1962).

[3]E. J. Powers in Ref. 1 (p. 161), and Phys. Fluids 8, 1155 (1965); W. Burton et al., Nucl. Fusion Suppl. 3, 903 (1962); M. Babykin et al., Zh. Eksperim. i Teor. Fiz. 43, 411 (1962) [translation: Soviet Phys.—JETP 16, 295 (1963)]; K. I. Thomassen, Phys. Rev. Letters 10, 80 (1963); A. Plyutto, Zh. Eksperim. i Teor. Fiz. 39, 1589 (1960) [translation: Soviet Phys.—JETP 12, 1106 (1961)]; and others.

[4]B. B. Kadomtsev, Plasma Turbulence (Academic Press, Inc., New York, 1965).

[5]S. Yoshikawa, Phys. Fluids 5, 1272 (1965); O. Buneman, Phys. Rev. 115, 503 (1959).

[6]M. S. Green, J. Chem. Phys. 22, 398 (1954); R. Kubo, J. Phys. Soc. Japan 12, 570 (1957).

[7]L. Spitzer, Phys. Fluids 3, 659 (1960).