[12]G. Kalbfleisch et al., Bull. Am. Phys. Soc. 12, 1134 (1967).

[13]Report of Liverpool-Oslo-Padua Collaboration, in Proceedings of the Heidelberg Conference on High Energy Physics and Elementary Particles, Heidelberg, Germany, 1967 (to be published).

[14]For $p\bar{p}\to\pi_1^+\pi_2^-\pi_3^+\pi_4^-$ we fit the expression $\partial^2\sigma/\partial m_{12}\partial m_{34}=PS(1+a_1(\rho_{12}+\rho_{34})+a_2(f_{12}+f_{34})+a_3\rho_{12}\rho_{34}+a_4(\rho_{12}f_{34}+\rho_{34}f_{12}))$. PS represents the phase space, ρ_{12}, ρ_{34}, f_{12}, f_{34} are Breit-Wigner cross sections for particles 1,2 or 3,4 forming ρ^0 or f^0 resonance; a_1-a_4 are the parameters fitted. Reflections have been neglected. For single resonance production we find $\sigma(\rho^0\pi\pi)=(0.90\pm0.38)$ mb and $\sigma(f^0\pi\pi)=(-0.05\pm0.20)$ mb. The analysis of Ref. 8 shows that the $\rho\pi\pi$ final state occurs mainly through A production. For double resonance production we find $\sigma(\rho^0\rho^0)=(0.18\pm0.11)$ mb and $\sigma(\rho^0f^0)=(0.84\pm0.17)$ mb constant over our momentum range. Similarly, for $p\bar{p}\to\pi_1^+\pi_2^-\pi_3^+\pi_4^-\pi_5^0$ we fit the expression $\partial^2\sigma/\partial m_{12}m_{345}=PS(1+a_1\rho_{12}+a_2A_{345}+a_3\rho_{12}A_{345})$, where we exclude the ω^0 mass region and use a Monte Carlo generated phase space accounting for that. A_{345} is a Breit-Wigner cross section for particles 3-5 forming A_2. For single resonance production we find $\sigma(\rho^0\pi\pi\pi)=0.93\pm0.15$ mb, $\sigma(A_2^0\pi\pi\pi)=0.52\pm0.15$ mb, $\sigma(\rho^+\pi\pi\pi)=0.86\pm0.14$ mb, and $\sigma(A_2^+\pi\pi\pi)=0.46\pm0.15$ mb. For double resonance production we find $\sigma(\rho^0A_2^0)=0.14\pm0.11$ mb and $\sigma(\rho^\pm A_2^+)=-0.07\pm0.10$ mb.

SEMICLASSICAL METHOD AND ZERO-POINT OSCILLATIONS

I. R. Senitzky
Institute for Exploratory Research, U. S. Army Electronics Command, Fort Monmouth, New Jersey
(Received 5 February 1968)

The interaction of two weakly coupled resonant systems is considered. It is shown that if one system is treated quantum mechanically and the other classically, the results that depend on the zero-point oscillations of the quantum-mechanical system cannot be expected to be physically meaningful.

The method of analyzing the interaction between two systems in which one system is treated classically and the other quantum mechanically—to which we refer as the semiclassical method—is as old as quantum mechanics. A well-known example of such a treatment is semiclassical radiation theory, in which the field quaned classically and matter quantum mechanically. The reverse practice of treating the field quantum mechanically and matter classically is also found in the literature, and there exist other applications of the semiclassical method in which neither system is the electromagnetic radiation field. Recently, there has been a spirited discussion about the validity of certain applications of semiclassical radiation theory to optics,[1] but the validity of other semiclassical treatments has received little examination. It is the purpose of the present Letter to point out limitations of the semiclassical method for general systems in weak resonant interaction. It turns out that the derived effect of the zero-point oscillations of the quantum mechanical system cannot be regarded as physically meaningful.

We consider two weakly coupled systems, each of which exhibits one or more natural frequencies and responds to a perturbation at frequency ω, the only common frequency, when it is in the ground state. Let the two systems be labeled "a" and "b," the respective Hamiltonians H_a and H_b, and the respective dynamical variables which exhibit the oscillations Q_a and Q_b. (We assume, for simplicity, that the resonant interaction takes place through single degrees of freedom.) The Hamiltonian for the combined system may be written as

$$H=H_a+H_b+\tfrac{1}{2}\gamma\hbar\{Q_a,Q_b\}, \tag{1}$$

where Q_a and Q_b are taken to be dimensionless and where the symmetrized product $\{Q_a,Q_b\}=Q_aQ_b+Q_bQ_a$ is used purely for later convenience, since $Q_a(t)$ and $Q_b(t)$ commute. The coupling constant γ is to be considered a small quantity of first order, and our analysis will be that of perturbation theory up to second order. The coupling is assumed to begin at $t=0$. The formalism will be such as to be interpretable both classically and quantum mechanically for each system independently, the dynamical variables being those of the Heisenberg picture for quantum mechanical interpretation, and $[A,B]$ representing the Poisson bracket of A and B multiplied by i for classical interpretation (with the canonically conjugate coordinates and momenta dimensionless). Nonresonant effects will be considered negligible compared with resonant effects. This approximation is illustrated by the

following equations:

$$\begin{aligned}\Delta H_a(t) &\equiv H_a(t)-H_a(0),\\ &= -\tfrac{1}{2}\gamma\hbar\int_0^t dt_1\{\dot{Q}_a(t_1), Q_b(t_1)\},\\ &\approx \tfrac{1}{2}\gamma\hbar\int_0^t dt_1\{Q_a(t_1), \dot{Q}_b(t_1)\},\\ &= -\Delta H_b(t).\end{aligned} \tag{2}$$

Indicating the perturbation-theory order by a superscript, we have

$$H_a^{(1)}(t) = -\tfrac{1}{2}\gamma\hbar\int_0^t dt_1\{\dot{Q}_a^{(0)}(t_1), Q_b^{(0)}(t_1)\}, \tag{3}$$

$$H_a^{(2)}(t) \approx -\tfrac{1}{2}\gamma\hbar\int_0^t dt_1[\{\dot{Q}_a^{(0)}(t_1), Q_b^{(1)}(t_1)\}-\{Q_a^{(1)}(t_1), \dot{Q}_b^{(0)}(t_1)\}]. \tag{4}$$

Using the relationship[2]

$$Q_a^{(1)}(t) = -i\gamma\int_0^t dt_1[Q_a^{(0)}(t), Q_a^{(0)}(t_1)]Q_b^{(0)}(t_1), \tag{5}$$

one obtains

$$\begin{aligned}H_a^{(2)}(t) = \tfrac{1}{2}i\gamma^2\hbar\int_0^t dt_1\int_0^{t_1} dt_2([Q_b^{(0)}(t_1), Q_b^{(0)}(t_2)]\{\dot{Q}_a^{(0)}(t_1), Q_a^{(0)}(t_1)\}\\ -[Q_a^{(0)}(t_1), Q_a^{(0)}(t_2)]\{Q_b^{(0)}(t_1), Q_b^{(0)}(t_2)\}).\end{aligned} \tag{6}$$

Consider now the condition under which either $\langle Q_a^{(0)}(t)\rangle_{\text{res}}$ or $\langle Q_b^{(0)}(t)\rangle_{\text{res}}$ vanishes, where by $\langle Q^{(0)}(t)\rangle_{\text{res}}$ we mean the part of the expectation value $\langle Q^{(0)}(t)\rangle$ that oscillates with frequency ω. Then

$$\langle H_a^{(1)}(t)\rangle = \langle H_b^{(1)}(t)\rangle = 0, \tag{7}$$

and

$$\langle \Delta H_a(t)\rangle = \tfrac{1}{2}i\gamma^2\hbar\int_0^t dt_1\int_0^{t_1} dt_2[C_b(t_1,t_2)S_a(t_1,t_2)-C_a(t_1,t_2)S_b(t_1,t_2)], \tag{8}$$

where

$$C(t_1,t_2)\equiv\langle[Q^{(0)}(t_1), Q^{(0)}(t_2)]\rangle, \quad S(t_1,t_2)\equiv\langle\{\dot{Q}^{(0)}(t_1), Q^{(0)}(t_2)\}\rangle.$$

We are particularly interested in C and S when the system to which these quantities refer is in the ground state. Quantum mechanically, with the matrix element notation,

$$Q_{ij}^{(0)}(t)\equiv Q_{ij}\exp(i\omega_{ij}t), \quad \hbar\omega_{ij}\equiv E_i-E_j, \tag{9}$$

we have, for the ground state ($E=E_0$),

$$\tilde{C}(t_1,t_2) = 2i\textstyle\sum_k |Q_{0k}|^2\sin\omega_{0k}(t_1-t_2), \quad \tilde{S}(t_1,t_2) = -2\sum_k \omega_{0k}|Q_{0k}|^2\sin\omega_{0k}(t_1-t_2), \tag{10}$$

a tilde indicating that the system is in the ground state. The resonant contribution of the products $\tilde{C}_b\tilde{S}_a$ and $\tilde{C}_a\tilde{S}_b$ will come only from the $k=r$ terms for which $\omega_{0r}=-\omega$; so we can write

$$\tilde{C}_{\text{res}}(t_1,t_2) = -2i|Q_{0r}|^2\sin\omega(t_1-t_2), \quad \tilde{S}_{\text{res}}(t_1,t_2) = -2\omega|Q_{0r}|^2\sin\omega(t_1-t_2). \tag{11}$$

It is immediately evident that the resonant contributions of the above products in the integrand of Eq. (8) cancel, and $\langle \Delta H_a \rangle = \langle \Delta H_b \rangle = 0$ when both systems (treated quantum mechanically) are in the ground state, as we would expect, of course.

Those systems which have the same formal Hamiltonian classically and quantum mechanically will also have the same formal expression for $[Q^{(0)}(t_1), Q^{(0)}(t_2)]$ classically and quantum mechanically, in terms of the dynamical variables of the system. It is clear, therefore, that for a classical system in the ground state, that is, in a condition of minimum energy, $\tilde{C}(t_1, t_2) \neq 0$ identically. This result likewise follows from the fact that if $\tilde{C}_b(t_1, t_2)$ vanished, the "b" system could not respond (up to second order) to an external, prescribed perturbation of frequency ω when in the ground state. Such a response to a prescribed force $q(t)$ with coupling Hamiltonian $\alpha \hbar Q_b q$ is expressed, in fact, by

$$\langle \Delta H_b(t) \rangle = -\tfrac{1}{2} i \alpha^2 \hbar \int_0^t dt_1 \int_0^{t_1} dt_2 \tilde{C}_b(t_1, t_2) \{\dot{q}(t_1), q(t_2)\}, \tag{12}$$

which also shows that the sign of $i\tilde{C}_b(t_1, t_2)$ must be such as to yield a positive value for the right-hand side of Eq. (12). As far as $\tilde{S}(t_1, t_2)$ is concerned, however, the situation is different. A classical system at minimum energy has no internal motion, and we have, classically, $\tilde{S}(t_1, t_2) = 0$. It is seen that for two classical systems in the ground state we obtain the expected result $\langle \Delta H_a \rangle = \langle \Delta H_b \rangle = 0$, as in the case of the quantum-mechanical systems. In this case, the result is due to the vanishing of $\tilde{S}$ for both systems.

The (universal) nonvanishing of $\tilde{S}$ for a quantum-mechanical system gives rise to zero-point oscillation—a strictly quantum-mechanical phenomenon intimately connected with the uncertainty principle—and $\tilde{S}$ may be regarded as a measure of this oscillation. It is seen from Eq. (8) that when two quantum-mechanical systems are coupled, the effects of the zero-point oscillation of both systems on the energy of either system cancel.

Consider now the situation in which the "a" system is treated quantum mechanically and the "b" system classically. Here we have $\tilde{C}_a$, $\tilde{C}_b$, and $\tilde{S}_a$ different from zero and $\tilde{S}_b$ equal to zero, so that

$$\langle \Delta H_b(t) \rangle = -\langle \Delta H_a(t) \rangle = -\tfrac{1}{2} i \gamma^2 \hbar \int_0^t dt_1 \int_0^{t_1} dt_2 \tilde{C}_b(t_1, t_2) \tilde{S}_a(t_1, t_2). \tag{13}$$

We see that when a classical system is coupled to a quantum-mechanical system, with both in the ground state, the formalism indicates that the classical system gains energy and the quantum-mechanical system loses energy! We also note, by comparing Eq. (13) with Eq. (12), that to the classical system the zero-point oscillation of the quantum-mechanical system appears as a classical, force-producing oscillation which can do work!

Now, obviously, a system in the ground state cannot—by definition—lose energy. The above result indicates, therefore, that the formulation of a problem in which there is a mutual interaction between a classical and a quantum-mechanical system contains a fundamental inconsistency, apparently because we are treating two coupled systems according to different principles. No physical significance can be attached to the formal effect of the zero-point oscillation, since it is this effect that produces the inconsistency. In those applications of the semiclassical method where the effect of the quantum-mechanical system on the classical system is ignored, this inconsistency does not arise, since the approximation of ignoring the effect of the "a" system on the "b" system amounts to dropping the $C_b S_a$ term in the integrand of Eq. (8),[3] so that it cannot contribute in Eq. (13). If the effect of the quantum-mechanical system on the classical system is of significance, however, results related to zero-point oscillations cannot be considered valid. The magnitude of error[4] will depend, of course, on the specific situation under study. Consider, for instance, the spontaneous emission power[5] of a two-level "atom," the "a" system, into a classical radiation mode, the "b" system. A simple calculation shows that S_a has the same value for both the upper and lower states, that is, the excited oscillations and the zero-point oscillation are of equal magnitude. The spontaneous emission power P_{sc} is therefore independent of the atomic state, and can be expressed easily in terms of the parameters of Eq. (8). Using Eq. (8) again for a fully quantum-mechanical treatment (indicated by the subscript qm), one obtains $P_{qm} = 2\rho_{11} P_{sc}$, where ρ is the

atomic density matrix and the indices 0, 1 indicate the lower and upper levels, respectively. Thus the relative error in P_{sc} varies from $\frac{1}{2}$ to ∞, depending on the atomic state, the greatest error occurring when the effect on the mode is entirely due to the atomic zero-point oscillations.[6] Another example of the application of the semiclassical method where the zero-point oscillations determine the result, and thus destroy its reliability, is the calculation of "quantum noise" (which is the amplified zero-point oscillation of the field) in a parametric amplifier[7]; in this case the field is treated quantum mechanically and the matter classically. Incidentally, the above discussion indicates that an intuitive picture—which is usually an essentially classical picture—of zero-point oscillations is likely to be misleading.

---

[1]See, for instance, L. Mandel and E. Wolf, Rev. Mod. Phys. 37, 231 (1965), where a comprehensive list of additional references is given.

[2]I. R. Senitzky, Phys. Rev. 155, 1387 (1967).

[3]This approximation has been used in treating optical detection. Thus, if the effect of the (detector) atoms on the field is ignored, a classical treatment of the field does not lead to any difficulties.

[4]By "error," we mean the deviation from a fully quantum-mechanical treatment.

[5]Spontaneous emission power of system "a" into system "b" is given by $-(d/dt)\langle H_a(t)\rangle$, from Eq. (8), with system "b" in the ground state.

[6]E. T. Jaynes and F. W. Cummings, Proc. IEEE 51, 89 (1963), suggest a modified form of the semiclassical method (which they call neoclassical theory) for application to the mutual interaction between atomic systems and a classical radiation mode. They propose that in those equations of motion which describe the effect of the atoms on the field, the current variables be replaced by their expectation values; while in those equations which describe the effect of the field on the atoms, the atoms remain fully quantum mechanical. According to this prescription, the (neoclassical) spontaneous emission P_{nc}, which can be obtained from Eq. (8) through the replacement of $\langle\{Q_a^{(0)}(t_1), Q_a^{(0)}(t_2)\}\rangle$ by $2\langle Q_a^{(0)}(t_1)\rangle\langle Q_a^{(0)}(t_2)\rangle$, is given by $P_{nc} = 2|\rho_{01}|^2 P_{sc} = (|\rho_{01}|^2/\rho_{11})P_{qm}$. The factor $|\rho_{01}|^2/\rho_{11}$ varies from 0 to 1 as the atomic state, taken to be a superposition of the two energy states, varies from the upper to the lower one. P_{nc} becomes equal to P_{qm} only when both vanish. The effect of zero-point oscillation is indeed eliminated by this method, but so is the effect of excited oscillation when the system is in the upper energy state. In fact, this method eliminates the effect of excited oscillation of any quantum-mechanical system whenever that system is in an energy state, no matter how high the energy level may be. Thus, in the case of spontaneous emission by a quantum-mechanical harmonic oscillator into the radiation field, the error introduced by the neoclassical method can be arbitrarily large; furthermore, this method is clearly inconsistent with the Correspondence Principle, according to which an oscillator in a high energy state corresponds to a classical excited oscillator with unpredictable phase. It should be noted that the above discussion does not apply to induced emission problems involving the effect of atoms on a classical field but unrelated to zero-point oscillation.

[7]W. H. Louisell, A. Yariv, and A. E. Siegman, Phys. Rev. 124, 1646 (1961); N. M. Kroll, in Quantum Optics and Electronics, edited C. DeWitt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach Publishers, Inc., New York, 1964), p. 19; B. R. Mollow and R. J. Glauber, Phys. Rev. 160, 1076 (1967).