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## SPIN-INDEPENDENT OSCILLATIONS OF A DEGENERATE ELECTRON LIQUID*

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We derive a dispersion relation valid for arbitrary wavelength of the spin-independent oscillations of a electron fluid. Analytic expressions for the long-wavelength limit can be obtained quite easily. In addition, results for all wavelengths can be obtained numerically for the situation in which the Fermi-liquid interaction is approximated by a finite series of spherical harmonics.

In his classic paper on the oscillations of a degenerate electron fluid in the presence of a dc magnetic field, Silin[1] considers both spin waves and spin-independent disturbances. For the spin-independent disturbances he proceeds by using Maxwell's equations to relate the electric field to the charge and current densities, and then expresses these in terms of the distribution function $f(\theta,\varphi)$. The kinetic equation then becomes a homogenous integral equation for the function $f(\theta,\varphi)$. Silin then expands $f(\theta,\varphi)$ in the form of a series of spherical harmonics, and notes that in the limit of infinite wavelength, the different harmonics are independent. For $n \geq |m| > 1$, he obtains the eigenfrequencies[2] $\omega_{nm} = n\omega_c(1+A_n)+O(q^2)$, where $\omega_c$ is the cyclotron frequency and $q$ is the wave vector. Recently, Mermin and Cheng[3] have extended Silin's analysis to shorter wavelength by evaluating the term of order $q^2$ for propagation perpendicular to the dc magnetic field. Essentially, these authors note that the $n,m$ spherical harmonic is connected by a term linear in $q$ to the $n\pm1, m\pm1$ spherical harmonics. They then use perturbation theory to evaluate the $q^2$ term.

In this note we present a method of analysis which is valid for arbitrary wavelength. The long-wavelength limit of our result reproduces, in a rather simple fashion, the results of Mermin and Cheng. In addition, if the Fermi-liquid interaction is approximated by a finite number of terms in the usual spherical-harmonic expansion, we need only solve a finite-size determinantal equation for any value of $qr_c$, where $r_c$ is the cyclotron radius. If all the coefficients $A_n$ are small compared with unity, then a calculation linear in the $A_n$ should be adequate for values of $qr_c > 1$, and the determinantal equation becomes rather simple. In particular, if all the $A_n$ are set equal to zero for $n \geq 2$, the secular equation reduces to the well-known dispersion relations $\sigma_{zz}{}^0 \approx 0$ and $\sigma_{xx}{}^0\sigma_{yy}{}^0 + \sigma_{xy}{}^{0^2} \approx 0$ for polarizations parallel and perpendicular to the dc field, respectively,[4] where $\sigma^0$ is the conductivity tensor in the absence of Fermi-liquid effects.

The spin-independent kinetic equation for a collisionless electron liquid is

$$(-i\omega + iq_x v + \omega_c \partial/\partial\varphi) f(\theta,\varphi) + (iqv_x + \omega_c \partial/\partial\varphi)\delta\epsilon_1(\theta,\varphi) + e\vec{E}\cdot\vec{v} = 0. \quad (1)$$

Here $\theta,\varphi$ are polar coordinates in $\vec{k}$ space and $f(\theta,\varphi)$ is defined by

$$\delta f(\vec{k}) = (-\partial f_0/\partial\epsilon) f(\theta,\varphi), \quad (2)$$

where $\delta f(\vec{k})$ is half the trace with respect to spin of the deviation from thermal equilibrium of the density matrix caused by the electric field $\vec{E}$. We have assumed space-time dependence of the form $\exp(-i\omega t + iq\cdot x)$, and taken the dc magnetic field to define the $z$ direction. The function $\delta\epsilon_1$ is given by

$$\delta\epsilon_1(\vec{k}) = \frac{2}{(2\pi)^3}\int d^3k'\, \Phi(\vec{k},\vec{k}')\delta f(\vec{k}'), \quad (3)$$

where $\Phi(\vec{k},\vec{k}')$ is the spin-independent part of the interaction function. We introduce $\vec{R}(\theta,\varphi)$, the periodic part of the position vector in real space of an electron on the Fermi surface, and note that

$$e^{i\vec{q}\cdot\vec{R}(\theta,\varphi)} = \sum_{m=-\infty}^{\infty} J_m(X\sin\theta)e^{im\varphi}. \quad (4)$$

Here $X = q_x v_F/\omega_c$ and $J_m$ is the $m$th-order Bessel function. We define the Fourier coefficients

$f_m(\theta)$, $v_m(\theta)$, and $\Phi_m(\theta,\theta')$ of the functions $fe^{i\vec{q}\cdot\vec{R}}$, $ve^{i\vec{q}\cdot\vec{R}}$, and $\Phi(\vec{k},\vec{k}')$, respectively, in analogy with Eq. (4). In terms of these functions, the kinetic equation can be written

$$(\omega-n\omega_c)f_n(\theta)-n\omega_c Y_n(\theta)-ie\vec{E}\cdot\vec{v}_n(\theta)=0. \tag{5}$$

Here

$$Y_n(\theta)=\sum_{ml}\sum_{s>|l|}A_s\Theta_s{}^l(\theta)J_{n-l}(X\sin\theta)\int d(\cos\theta')\Theta_s{}^l(\theta')J_{m-l}(X\sin\theta')f_m(\theta'), \tag{6}$$

and

$$\Theta_s{}^l(\theta)=\left[(s+\tfrac{1}{2})\frac{(s-|l|)!}{(s+|l|)!}\right]^{1/2}P_s{}^{|l|}(\cos\theta). \tag{7}$$

$P_s{}^{|l|}$ is the standard Legendre polynomial, and the $A_s$ are the Fermi-liquid interaction coefficients.[4] Following Silin, we use Maxwell's equations to relate the electric field to the current density and hence to the distribution function. We obtain

$$\vec{E}=+\frac{4\pi i}{\omega}\vec{\Lambda}\cdot\left(\frac{em^*p_F}{2\pi^2}\right)\gamma_1\sum_n\int d(\cos\theta)f_n(\theta)\vec{v}_n{}^*. \tag{8}$$

Here $\vec{\Lambda}$ is a diagonal matrix with an $x$-$x$ element of unity and the other diagonal elements equal to $\omega^2/(\omega^2-c^2q^2)$, and $\gamma_n=1+A_n$. We introduce the function

$$\Gamma_n{}^m=\sum_s\int d(\cos\theta)\Theta_n{}^m(\theta)f_s(\theta)J_{s-m}(X\sin\theta). \tag{9}$$

The kinetic equation can be solved[5] by substituting Eq. (8) into Eq. (5), multiplying by $\Theta_{s'}{}^{l'}(\theta)J_{n-l'}(X\times\sin\theta)$, summing on $n$, and integrating over $\cos\theta$. When this is done one obtains an infinite set of simultaneous equations for the function $\Gamma_{s'}{}^{l'}$. Now, not all the $\Gamma_n{}^m$ are independent.[6] The equation of continuity gives

$$\Gamma_0{}^0=6^{-1/2}(qv_F/\omega)\gamma_1(\Gamma_1{}^1+\Gamma_1{}^{-1}). \tag{10}$$

We can use Eq. (10) to eliminate $\Gamma_0{}^0$ from the infinite set of equations. Then the solutions are obtained by setting the determinant of the coefficients equal to zero. We find it most convenient to write the infinite determinantal equation as

$$|a_{nn'}{}^{mm'}-\delta_{nn'}{}^{mm'}|=0, \tag{11}$$

where for $n',n\neq 1$,

$$a_{nn'}{}^{mm'}=-A_{n'}\delta_{nn'}{}^{mm'}+\sum_s\frac{\omega A_{n'}}{\omega-s\omega_c}\int d(\cos\theta)\Theta_n{}^m J_{s-m}\Theta_{n'}{}^{m'}J_{s-m'}; \tag{12}$$

while for $n'=1,n\neq 1$,

$$a_{n1}{}^{mm'}=\sum_s\frac{\omega}{\omega-s\omega_c}\Lambda_1{}^{m'}\int d(\cos\theta)\Theta_n{}^m J_{s-m}\Theta_1{}^{m'}u_s{}^{m'}; \tag{13}$$

for $n=1,n'\neq 1$,

$$a_{1n'}{}^{mm'}=\sum_s A_{n'}\left(-1+\frac{\omega}{\omega-s\omega_c}\right)\int d(\cos\theta)\Theta_1{}^m u_s{}^m\Theta_{n'}{}^{m'}J_{s-m'}; \tag{14}$$

and for $n=n'=1$,

$$a_{11}{}^{mm'}=\sum_s\frac{\omega}{\omega-s\omega_c}\Lambda_1{}^{m'}\int d(\cos\theta)\Theta_1{}^m u_s{}^m\Theta_1{}^{m'}u_s{}^{m'}. \tag{15}$$

In these equations we have introduced

$$u_s^{\pm 1} = J_{s-1} \pm J_{s+1}, \quad u_s^{0} = J_s, \tag{16}$$

and $\Lambda_1^{-1} = \frac{1}{2}\omega_p^2\gamma_1/(\omega^2 - c^2q^2)$, $\Lambda_1^{0} = \omega_p^2\gamma_1/(\omega^2 - c^2q^2)$, and $\Lambda_1^{+1} = \frac{1}{2}\omega_p^2\gamma_1/\omega^2$. In writing down Eqs. (13) and (15) we have omitted small terms involving $A_0$ and $A_1$ compared with the very large terms proportional to $\Lambda_1^{1}$, $\Lambda_1^{-1}$, or $\Lambda_1^{0}$ for $\omega$ of order $\omega_c$. Because of the appearence of the large terms coming from Maxwell's equation, $A_0$ and $A_1$ essentially drop out of the problem. In addition, we can neglect the $-1$ compared with $a_{11}^{11}$, $a_{11}^{00}$, and $a_{11}^{-1-1}$ in Eq. (11).

It is not difficult to see that $a_{nn'}^{mm'}$ vanishes if $n-m$ is of different parity from $n'-m'$. Thus the secular equation reduces to one for odd $n-m$ and one for even $n-m$. It can easily be seen that all the off-diagonal elements of $a_{nn'}^{mm'}$ vanish in the $q=0$ limit except $a_{11}^{1-1}$ and $a_{11}^{-11}$. The solutions become

$$\omega = \gamma_n m\omega_c + \alpha_{nm}X^2 \tag{17}$$

for $n>1$ and $|m|<n$. The coefficient $\alpha_{nm}$ can be evaluated by noting that for the $n,m$ mode, $a_{nn}^{mm} - 1$ is proportional to $X^2$; and that only the matrix elements involving $n\pm 1, m\pm 1$ are proportional to $X$. For $n>2$ we obtain the result

$$\alpha_{nm} = \frac{-\omega_c\gamma_n}{2}\left\langle {m \atop n} \middle| \sin^2\theta \middle| {m \atop n} \right\rangle \frac{mA_n}{1-m^2A_n^2} + \frac{m\omega_c\gamma_n^2}{4}\sum_{n'm'}{}' \frac{An'}{m\gamma_n - m'} \left\langle {m' \atop n'} \middle| \sin\theta \middle| {m \atop n} \right\rangle^2 (m\gamma_n - m'\gamma_{n'})^{-1}, \tag{18}$$

where the sum on $n',m'$ includes the terms $m'=m\pm 1$, $n'=n\pm 1$, only. For $n=2$, the terms involving $A_{n-1}$ must be replaced by

$$-\tfrac{1}{4}\omega_c\langle {}^{0}_{1}|\sin\theta|{}^{1}_{2}\rangle^2 \text{ for } m=1$$

and by

$$-\frac{1}{4}\frac{\gamma_2\omega_c}{2\gamma_2-1}\langle {}^{1}_{1}|\sin\theta|{}^{2}_{2}\rangle^2 \text{ for } m=2.$$

In these equations

$$\left\langle {m \atop n} \middle| \sin\theta \middle| {m' \atop n'} \right\rangle = \int d(\cos\theta)\Theta_n^{m}\sin\theta\Theta_{n'}^{m'}. \tag{19}$$

These results are equivalent to those obtained by Mermin and Cheng[3] using perturbation theory.

When the interaction function is approximated by a finite series of spherical harmonics, the secular equation is finite because $a_{nn'}^{mm'}$ is proportional to $A_{n'}$ for $n'>1$. We have solved the secular equation numerically[5] for the case $A_n=0$ for $n>2$. For sufficiently large values of $X$ ($X>1$), linearization in the interaction coefficients $A_n$ is a good approximation, and the secular equations become

$$a_{11}^{00}\Big(1-\sum_{nm}a_{nn}^{mm}\Big)+\sum_{nm}a_{1n}^{0m}a_{n1}^{m0}=0 \tag{20}$$

for polarization parallel to the dc field, and

$$K_0\Big[1-\sum_{nm}a_{nn}^{mm}\Big]+a_{11}^{11}\sum_{nm}a_{n1}^{m-1}a_{1n}^{-1m}+a_{11}^{-1-1}\sum_{nm}a_{n1}^{m1}a_{1n}^{1m} - a_{11}^{-11}\sum_{nm}a_{n1}^{m-1}a_{1n}^{1m}-a_{11}^{1-1}\sum_{nm}a_{n1}^{m1}a_{1n}^{-1m}=0 \tag{21}$$

for the other polarization. In these equations the summations are performed over all $n>1$ and all $m$ such that $|m|<n$; in addition, $n-m$ must be odd in Eq. (20) and even in Eq. (21). The function $K_0$ is equal to $a_{11}^{11}a_{11}^{-1-1} - a_{11}^{-1,1}a_{11}^{1,-1}$. On setting all the $A_n=0$, Eqs. (20) and (21) reduce to $a_{11}^{00}=0$ and $K_0=0$, respectively. These are identical to the equations $\sigma_{zz}=0$ and $\sigma_{xx}\sigma_{yy}+\sigma_{xy}^2=0$ for the noninter-

acting electron gas. The $q=0$ roots at $\omega=m\omega_c$ found from these equations are actually the zero-sound modes $\omega=\gamma_n m\omega_c$ (for $n>1, n>|m|$) which all coalesce when $\gamma_n=1$ for all values of $n$. This can be seen by substituting back into Eq. (11) and noting that $\Gamma_1^m$ vanishes for these modes[7]; thus there is no electric field or current density associated with them at $q=0$. When all the $A_n$ are finite, the solution for $n=1$ is the transverse or longitudinal plasma frequency. The "plasma waves" observed by Walsh and Platzman[4] are predominantly zero-sound waves with $m=1$ for small values of $X$. For a given $m$, the "zero-sound" wave with the smallest possible value of $n$ is most strongly coupled to an electric field for long wavelengths. Thus for $m=1$, the $n=2$ mode is most strongly excited for polarization parallel to the dc field, and the $n=3$ mode is for the other polarization.

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[6]Silin does not make use of the equation of continuity to eliminate the $n=0, m=0$ harmonic from the secular equation.

[7]It should be noted that, despite the rather complicated definitions [Eq. (10)] of $\Gamma_n^m$, these functions turn out to be identical to the spherical harmonics used by Silin.

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## TRANSFERRED HYPERFINE INTERACTION OF ISOELECTRONIC $Yb^{3+}$ AND $Tm^{2+}$ IN $CaF_2$

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ENDOR measurements of the $^{19}F$ nuclei in the first four shells in $CaF_2$ containing $Yb^{3+}$ ions in the cubic site are reported. The results are compared with those of isoelectronic $Tm^{2+}$ ions.

Transferred hyperfine interactions in rare earths have been measured so far only in a few cases.[1,2] The interpretation of these results is, to date, not very conclusive because the various different contributions such as overlap, covalency, dipolar interaction, and configuration interaction are difficult to evaluate. Some light could be shed on this problem if isoelectronic systems would be studied, in which the paramagnetic ion is contained in the same crystal host. This possibility exists for the alkaline earth fluorides such as $CaF_2$, $SrF_2$, or $BaF_2$. It has been shown by numerous authors that stable divalent rare-earth ions can be produced in this crystal host,[3] where the rare-earth ions take the place of the cation and preserve the cubic symmetry. On the other hand, trivalent rare-earth ions can be substituted for the calcium ion. A fraction of these ions, which depends of the nature of the ion —the manner in which the crystals were grown and the subsequent heat treatment—can be substituted at the cubic site with charge compensation far removed.

We have studied the ENDOR fluorine spectrum of the first four shells in the single crystal of $CaF_2$ containing approximately 0.1% of $Yb^{3+}$ ions. The results are contained in Table I. They are to be compared with Bessent and Hayes' measurements on $Tm^{2+}$ in $CaF_2$.[2] Also listed in Table I are the $g$ factors, the cubic crystal-field parameters as determined from the optical spectra,[4,5] and the deviation of the $g$ factor from the