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MAGNITUDE OF RENORMALIZATION CONSTANTS*

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Five years ago Källén[1] published a proof that at least one of the renormalization constants in quantum electrodynamics is infinite. A subsequent more refined analysis[2] led to the conclusion that it is the electron wave function renormalization which diverges. Recently Johnson[3] raised the question whether a formulation of the theory in another gauge might not alter this result. We have re-examined this problem and arrive at the conclusion that (a) Källén's proof is not conclusive, (b) the renormalization constants could be finite under rather special circumstances, and (c) the question of gauge invariance is quite irrelevant to this problem. We begin with a brief review of Källén's work on this problem.[4] It can be shown that the charge renormalization $(1-L)^{-1/2}$ can be expressed in the form

$$(1-L)^{-1}=1+\overline{\Pi}(0), \tag{1}$$

where by definition

$$\overline{\Pi}(Q^2)=P\int_0^\infty da\,\frac{\Pi(-a)}{a+Q^2}, \tag{2}$$

and

$$\Pi(Q^2)=-\frac{V}{3Q^2}\sum_{p^{(n)}=Q}\langle 0|j_\mu|n\rangle\langle n|j_\mu|0\rangle; \tag{3}$$

P denotes the principal value. Källén shows that in spite of the indefinite metric associated with quantum electrodynamics, $\Pi(-a)$ is positive for positive a, and that therefore a lower limit to the integral in Eq. (1) can be obtained by considering any subset of eigenstates of the Hamiltonian. The simplest of these are the states consisting of an incoming electron-positron pair. Our task therefore is to study the high-energy limit of the matrix element $\langle 0|j_\mu|p,p'\rangle$.

By the use of reduction formulas[1,5] one may obtain the following compact expression for the matrix element in question:

$$\langle 0|j_\mu|p,p'\rangle=\langle 0|j_\mu^{(0)}|p,p'\rangle[1-\overline{\Pi}(Q^2)+\overline{\Pi}(0)-i\pi\Pi(Q^2)+F_1(Q^2)-F_1(0)]+iQ_\nu\langle 0|m_{\mu\nu}^{(0)}|p,p'\rangle F_2(Q^2), \tag{4}$$

$$Q=p+p', \tag{4a}$$

where

$$N^2\bar{u}(p')\{\iint d^4x\,d^4y\,e^{i(p'x+py)}[\theta(-x)\theta(-y)\langle 0|\{[j_\mu(0),\bar{f}(x)],f(y)\}|0\rangle-\theta(-y)\theta(y-x)\langle 0|\{[j_\mu(0),f(y)],\bar{f}(x)\}|0\rangle]\}v(p)$$

$$=F_1(Q^2)\langle 0|j_\mu^{(0)}|p,p'\rangle+iF_2(Q^2)Q_\nu\langle 0|m_{\mu\nu}^{(0)}|p,p'\rangle. \tag{5}$$

In terms of Källén's notation, $F_1(Q^2)$ and $F_2(Q^2)$ are given by[6]

$$F_1(Q^2)+F_2(Q^2)=\bar{R}^{\rm reg}(Q^2)+i\pi R^{\rm reg}(Q^2),$$
$$F_2(Q^2)=\bar{S}^{\rm reg}(Q^2)+i\pi S^{\rm reg}(Q^2). \tag{6}$$

Here N stands for the electron wave function renormalization constant and $f(x)$ is the fermion source density. The matrix elements with the superscript (0) are the lowest order perturbation expressions for the current and the spin. In this notation Ward's identity takes the form

$$F_1(0)=(1-N^2)/(1-L). \tag{7}$$

To estimate the behavior of this matrix element as $-Q^2\to\infty$, Källén argues as follows: let us assume that all the renormalization constants are finite. In particular this means that $\bar{\Pi}(0)$ is finite, so that $\Pi(-a)$ goes to zero fast enough as $a\to\infty$. For any reasonable[7] function $\Pi(-a)$, this implies that the same must hold for $\bar{\Pi}(Q^2)$. The problem therefore reduces to a study of the functions $F_1(Q^2)$ and $F_2(Q^2)$.

Källén originally examined the left-hand side of Eq. (5), and reduced it to an integral of the form

$$\int_{-\infty}^{\infty}\int dx\,dy\,\frac{F_\mu(\vec{p},x;\vec{p}',y)}{[x+y-(p_0+p_0'-i\epsilon)][y-p_0'+i\epsilon]}. \tag{8}$$

As $(p_0+p_0')\to\infty$ this integral need not vanish, because $\vec{p}$ and $\vec{p}'$ also vary when p_0 and p_0' vary (since $\vec{p}^2+m^2=p_0^2$; $\vec{p}'^2+m^2=p_0'^2$). The integral would vanish if the function $F_\mu(\vec{p},x;\vec{p}',y)$ approached zero as $\vec{p},\vec{p}'\to\infty$ but in general there is no way of establishing this. As the vanishing of this integral was an important part of Källén's original proof, we conclude that the proof is inconclusive.

A more constructive discussion can be given if we abandon a certain degree of generality, and assume that the functions $F_1(Q^2)$ and $F_2(Q^2)$ are analytic in the $-Q^2$ plane except for a cut along the positive real axis.[8] The behavior of the functions $F_1(Q^2)$, $F_2(Q^2)$ for large values of $-Q^2$ is unknown. The locality of the theory demands that they do not increase faster than a polynomial. In general we can therefore write

$$F_i(Q^2)=\frac{1}{\pi}\int_0^\infty da\,\frac{\rho_i(-a)}{a+Q^2-i\epsilon}+p_n^{(i)}(Q^2), \tag{9}$$

or, more relevantly,

$$\bar{R}^{\rm reg}(Q^2)=\bar{R}^{\rm reg}(0)+(+Q^2)\bar{R}^{\rm reg\prime}(0)+\ldots+\frac{(+Q^2)^{n-1}}{(n-1)!}\bar{R}^{{\rm reg}(n-1)}(0)$$
$$+(-Q^2)^n P\int_0^\infty\frac{da}{a^n}\,\frac{R^{\rm reg}(-a)}{a+Q^2}, \tag{10}$$

and $\bar{S}^{\rm reg}(Q^2)$ is given by a similar expression. The polynomials are seen to be real. It is important to note that the degree of the polynomial cannot be smaller but <u>may be larger</u> than the power necessary to make the integral over the imaginary part convergent.[9] Let us now discuss a few possibilities. We shall talk about R but the analysis applies equally well to S.

Case I. $n=0$.—In this case we have

$$\bar{R}^{\rm reg}(Q^2)=P\int_0^\infty da\,\frac{R^{\rm reg}(-a)}{a+Q^2}. \tag{11}$$

This implies that

$$\bar{R}^{\rm reg}(Q^2)-\bar{R}^{\rm reg}(0)=-\int_0^\infty\frac{da}{a}\,\frac{R^{\rm reg}(-a)}{1+a/Q^2}, \tag{12}$$

so that

$$\lim_{-Q^2\to\infty}\bar{R}^{\rm reg}(Q^2)=\bar{R}^{\rm reg}(0)-\int_0^\infty\frac{da}{a}\,R^{\rm reg}(-a)=0. \tag{13}$$

If $\bar{S}^{\rm reg}(Q^2)$ has the same behavior, we obtain [using Eq. (7)] the result

$$\langle 0|j_\mu|p,p'\rangle\to\frac{N^2}{1-L}\langle 0|j_\mu^{(0)}|p,p'\rangle, \tag{14}$$

i.e., the matrix element approaches the unrenormalized Born approximation, and a contradiction with the initial assertion that all renormalization constants are finite is obtained because $\Pi(-a)\to{\rm const}\,N^2/(1-L)^2$ as $a\to\infty$. This corresponds to the case discussed by Källén.

Case II. $n=1$.—In this case we have

$$\bar{R}^{\text{reg}}(Q^2)=\bar{R}^{\text{reg}}(0)+(-Q^2)\int_0^\infty \frac{da}{a}\,\frac{R^{\text{reg}}(-a)}{a+Q^2}. \quad (15)$$

If the integral

$$\int_0^\infty \frac{da}{a}\,R^{\text{reg}}(-a) \quad (16)$$

converges, we have

$$\lim_{-Q^2\to\infty} \bar{R}^{\text{reg}}(Q^2)=\bar{R}^{\text{reg}}(0)-\int_0^\infty da\,\frac{R^{\text{reg}}(-a)}{a}. \quad (17)$$

The difference between this case and the one discussed before is that the right side of Eq. (17) need not vanish. In particular, if this constant is such as to make $\langle 0|j_\mu|p,p'\rangle\to 0$ as $-Q^2\to\infty$, (this necessitates that there be no polynomial in the S^{reg} equation), then no conclusion as to the magnitude of the renormalization constants can be drawn from the electron-positron pair states alone. This is the only possible exception to Källén's result. If the integral in Eq. (16) does not converge, $\bar{R}^{\text{reg}}(Q^2)$ will diverge in the limit $-Q^2\to\infty$ for any reasonable function $R^{\text{reg}}(-a)$. Thus except for the special case noted above, the integral in Eq. (2) will always diverge (this is even more easily seen if $n=2, 3, \ldots$).[10] The special case corresponds to the "no-subtraction" spectral representations for the matrix elements of the current operator discussed recently.[11]

It is clear from this discussion that developing the theory in another gauge cannot make any difference. We have used gauge invariance only in the particular form in Eq. (5), but the scalar functions $F_1(Q^2)$ and $F_2(Q^2)$ can be discussed independently of whether the photon has a mass or not. This fact will affect the low-frequency behavior of the functions [e.g., Eq. (7)] but cannot make any difference in the high-energy region. Our negative conclusion applies by the same token to all renormalizable theories: one cannot prove that the renormalization constants are infinite, because there is always one possibility that a special cancellation occurs.

We may ask whether it is possible that the renormalization constants are indeed finite. For this to be true all matrix elements in Eq. (3) must have the "no-subtraction" form. Actually this is not enough, because even though every contribution to the integral $\bar{\Pi}(0)$ will be finite, it is not necessarily true that the sum of these contributions converges. In fact, since $\langle 0|j_\mu|n\rangle$ could not be expected to approach zero until $a>>(E_0N)^2$ (where E_0 is some fixed energy and N is the number of particles in the state), a very nonuniform series is involved, and counterexamples to convergence can readily be constructed.[12] Thus unless field theory has some regularity properties quite unsuspected from perturbation theory, the renormalization constants will be infinite.

In our discussion we have given up a certain amount of generality by working with the unproved spectral representations. Should these turn out to be wrong, there may be two possibilities: If the violation of analyticity is limited to a finite part of the plane (e.g., longer cuts), then the domination of the high-energy behavior by a polynomial is unaltered and our discussion in terms of spectral representations needs only minor modifications. This would not be true, however, if regions of nonanalyticity extend to infinity (e.g., cuts not indicated by perturbation theory such as those which might arise from complex "ghosts").

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[6] The superscript "reg" refers to a particular prescription for calculating these functions. This complication introduced by the necessity of adiabatic decoupling is of no import in the following argument, so that we will not take the space to discuss it, but rather refer the interested reader to reference 4.

[7] This point, summarized here so cavalierly, is discussed in detail in the appendix of reference 1.

[8] This property has been established to all orders in perturbation theory [Y. Nambu, Nuovo cimento 6, 1064 (1957)], and under certain circumstances, unrealistic for this theory, independent of perturbation theory [see Bremermann, Oehme, and Taylor, Phys. Rev. 109, 2178 (1958)].

[9] This was overlooked in an analysis of the problem

in terms of spectral representations given by G. Källén in Proceedings of the CERN Symposium on High-Energy Accelerators and Pion Physics, Geneva, 1956 (European Organization of Nuclear Research, Geneva, 1956), Vol. II, p. 187.

[10]Källén remarked in his paper that a divergence of $\lim_{-Q^2 \to \infty} \bar{R}^{\text{reg}}(Q^2)$ as well as its vanishing will lead to infinite renormalization constants, but he overlooked the only possible counterexample.

[11]S. Drell and F. Zachariasen, Phys. Rev. 111, 1727 (1958); M. Gell-Mann and J. Mathews (to be published).

[12]See reference 9. The infrared divergence in quantum electrodynamics implies a nonuniform behavior of this type for the series in Eq. (3). According to arguments of Bloch-Nordsieck type, the matrix element for producing an electron pair plus any finite number of photons should vanish. Nevertheless, the sum over all numbers of photons should yield a finite result.

ERRATUM

PRECISE DETERMINATION OF THE MUON MAGNETIC MOMENT. R. L. Garwin, D. P. Hutchinson, S. Penman, and G. Shapiro [Phys. Rev. Letters 2, 213 (1959)].

It has recently been[1] established that the theoretical uncertainties in the lower limit of the muon mass from mesonic α-rays[2] are much smaller than previously supposed.[3] We therefore undertook a more careful search into possible systematic errors in our moment measurement. We have found and corrected one serious source of error. A method of testing was then designed which simulated the experimental situation and permitted the establishment of an upper limit on the remaining systematic errors in the electronic equipment of 1×10^{-5} or almost an order of magnitude less than our stated accuracy.

The circuit design used in this experiment has been intended to be completely aperiodic, i.e., should introduce no systematic errors in a frequency measurement (as opposed to a random error in measuring each particle due to the finite time resolution). Preceding the analysis equipment, however, is the "zero-crossing detector" which does the fast timing of both the muon and decay electron pulses. To circumvent the difficulties of long-term transit-time variations in phototubes, a single counter and timing circuit was used for both muon and electron pulses. This raises the possibility that the recovery of the phototube or fast timing circuit after the passage of a muon pulse will alter the apparent time of the electron. This effect is important only if it varies during the measurement interval. In particular, if electrons immediately following the muon pulse are delayed more than those coming later in the gate interval, the net effect is to make the apparent muon precession frequency appear larger than it actually is. Unfortunately, we observed on our circuit diagram a one-microsecond time-constant at the grid following the distributed zero-crossing amplifier. Had this time-constant been 50 millimicroseconds or >20 microseconds, no error would have been incurred, but we estimated the maximum error caused by it as $\sim 4 \times 10^{-4}$.

Since there is available to us no measuring equipment that approaches the experimental apparatus in resolving ability, the existence of the postulated effect was established by inserting two pulses with fixed cable delay of 2 microseconds into the equipment. The first was analyzed by the circuitry as a muon pulse and the other as an electron pulse. The equipment output was displayed on a pulse-height analyzer as previously described. A third pulse which had been traveling in a long cable was then inserted between the "muon" and "electron" pulses. Its retarding effect on the electron timing could then be seen clearly. The fast timing circuit was then altered (by changing the offending time-constant) so that the effect was no longer apparent.

After modifying the circuitry, the following rigorous test was made to establish that there remained no other such sources of error. We built a source of random pulses whose probability of occurrence above a fixed threshold oscillated in time at a known rate. This consisted of a 6810A photomultiplier viewing a plastic scintillator which was irradiated with β rays. The focusing voltage of the tube was modulated at or near the frequency of the reference oscillator of the measuring apparatus, (86.2 Mc/sec). A 30-volt peak-to-peak signal was sufficient to achieve nearly 100% modulation of the output pulses. The slow coincidence circuits select the first pulse above a threshold after a dead time as a μ pulse and the next pulse as an electron. Since the probability of occurrence of these two