PRECESSION OF THE POLARIZATION OF PARTICLES MOVING IN A HOMOGENEOUS ELECTROMAGNETIC FIELD*

V. Bargmann
Princeton University, Princeton, New Jersey

Louis Michel
Ecole Polytechnique, Paris, France

and

V. L. Telegdi
University of Chicago, Chicago, Illinois

(Received April 27, 1959)

The problem of the precession of the "spin" of a particle moving in a homogeneous electromagnetic field – a problem which has recently acquired considerable experimental interest – has already been investigated for spin $\frac{1}{2}$ particles in some particular cases.[1] In the literature the results were derived by explicit use of the Dirac equation, with the occasional inclusion of a Pauli term to account for an anomalous magnetic moment. On the other hand, following a remark of Bloch[2] in connection with the nonrelativistic case, the expectation value of the vector operator representing the "spin" will necessarily follow the same time dependence as one would obtain from a classical equation of motion. To solve the problem for arbitrary spin in the relativistic case, it will thus suffice to produce a consistent set of covariant classical equations of motion. Such equations have been indicated a long time ago by Frenkel[3] and are discussed by Kramers.[4] These authors use an antisymmetric tensor M as the relativistic generalization of the intrinsic angular momentum observed in the rest-frame of the particle. A formulation in terms of the (axial) four-vector s which describes the polarization in a covariant fashion[5] – though basically equivalent – is however much more convenient for our problem. We shall therefore derive first the equations of motion directly in terms of this four-vector s.

Let the spin of the particle be represented[6] in the rest-frame (R) by $\vec{s}$. We assume (a) that there exists a four-vector s such that in (R) it coincides with $\vec{s}$:

$$s = (s^0, \vec{s}); \quad \text{in } (R),\ s = (0, \vec{s}). \tag{1}$$

Denoting the four-velocity of the particle by $u = (u^0, \vec{u}) = \gamma(1, \vec{v})$ [where $\vec{v}$ is the ordinary velocity, and $\gamma(v) = (1 - v^2)^{-1/2}$], one has in every frame

$$s \cdot u = 0, \text{ i.e., } s^0 = \vec{s} \cdot \vec{v}. \tag{2}$$

We further assume (b) that $\vec{s}$ obeys in (R) the customary equation of motion

$$d\vec{s}/d\tau = (ge/2m)(\vec{s} \times \vec{H}), \qquad (R) \tag{3}$$

where $\vec{H}$, e, and m have their standard meanings, while the gyromagnetic ratio g is defined by this very equation. While s^0 vanishes by hypothesis in any instantaneous rest-frame, $ds^0/d\tau$ need not. In fact, (2) implies

$$ds^0/d\tau = \vec{s} \cdot (d\vec{v}/d\tau), \qquad (R) \tag{4}$$

for such frames. In general, $du/d\tau = f/m$ (where f = four-force), while in a homogenous external electromagnetic field specified by $F = -(\vec{E}, \vec{H})$

$$du/d\tau = (e/m)F \cdot u. \tag{5}$$

The immediate generalization of Eqs. (3) and (4) to arbitrary frames is

$$ds/d\tau = (ge/2m)[F \cdot s + (s \cdot F \cdot u)u] - [(du/d\tau) \cdot s]u, \tag{6}$$

as can be checked by reducing to the rest-frame. With (5), one has for homogeneous fields

$$ds/d\tau = (e/m)[(g/2)F \cdot s + (g/2 - 1)(s \cdot F \cdot u)u]. \tag{7}$$

(5) and (7) constitute, for any value of g and arbitrary spin $\mathcal{S}$, a consistent set of equations of motion; they imply that $s \cdot s$ and $s \cdot u$ are constant, so that condition (2) is maintained.[7] For experiments of current interest, the main use of (7) is in the computation of the rate Ω at which longitudinal polarization is transformed into a transverse one (and vice versa). For this, we express s in the laboratory frame (L) in terms of two unit polarization four-vectors, e_l and e_t:

$$s/\mathcal{S} = e_l \cos\phi + e_t \sin\phi,$$

where

$$\mathcal{S} = (-s \cdot s)^{1/2},$$

$$e_l = \gamma(v, \vec{v}/v) \equiv \gamma(v, \hat{v}), \quad e_t = (0, \hat{n}),$$

$$\hat{n} \cdot \hat{n} = 1, \quad \hat{n} \cdot \hat{v} = 0. \qquad (L) \tag{8}$$

Clearly, $\Omega = d\phi/dt = d\phi/\gamma d\tau$. Introducing (8) into (7), and expressing all quantities as ordinary vectors, we find

$$\Omega = (e/m)\{(\vec{E}\cdot\hat{n}/v)[(g/2 - 1) - g/2\gamma^2] + (\hat{v}\cdot\vec{H}\times\hat{n})(g/2 - 1)\}. \quad (9)$$

The relevant "anomaly" of spin-$\frac{1}{2}$ particles, $(g/2 - 1)$, is clearly exhibited in (9) although our derivation was classical throughout.

We now specialize (9) to some cases of practical interest (the references are to experiments):

(A) $\vec{E}\times\vec{v} = \vec{H}\times\vec{v} = 0$; $\Omega = 0$. The character of the polarization does not change, but the transverse polarization precesses around $\vec{v}$ in longitudinal fields with an angular frequency $\omega = (ge/2m\gamma)H = (g/2)\omega_L$, as follows readily from (8).

(B)[8] $\vec{H}\cdot\hat{n}\times\hat{v} = H$, $\vec{E} = 0$; $\Omega = \omega_L(g/2 - 1)\gamma$, where ω_L is the Larmor frequency defined in (A) above.

(C)[9] $\vec{E}\cdot\hat{n} = E$, $\vec{H} = 0$; $\Omega = \omega_p[-g/2\gamma + (g/2 - 1)\gamma]$, where $\omega_p = eE/m\gamma v$ is the angular frequency of the particle's motion in the laboratory.

(D)[10] $\vec{E}\cdot\vec{H} = 0$, rectilinear motion: $\vec{E} = -\vec{v}\times\vec{H}$; $\vec{H}\cdot\hat{n}\times\hat{v} = H$, $\Omega = \omega_L(g/2\gamma)$.

(E)[11] $\vec{E}\cdot\vec{H} = 0$, $\vec{H}\cdot\hat{n}\times\hat{v} = H$, $\vec{E}\cdot\hat{n} = -Ev_x/v$, trochoidal motion: $E/H << 1$; $\Omega = (e/m)[(g/2 - 1)(H - Ev_x) + Ev_x/(\gamma^2 - 1)]$,

$$(\Delta\phi/2\pi) \text{ per loop} = \gamma(E/H)\gamma(v)[1 - (E/H)v_x](g/2 - 1) = \gamma(v')(g/2 - 1),$$

where v' is velocity in a frame where $E' = 0$.

The generalization of (6) to cover particles having an intrinsic electric dipole moment $\vec{\epsilon} = (g'e/2m)\vec{s}$ may be of interest. In the (R) frame, the effect of $\vec{\epsilon}$ is taken into account by adding $\vec{\epsilon}\times\vec{E}$ to the right-hand side of (3), while leaving (4) unchanged. Thus the required change in the right-hand side of (6) is the addition of a term $-(g'e/2m)[(F^*\cdot s) + (s\cdot F^*\cdot u)u]$, denoting by F^* the dual of F, i.e., $F^* = -(\vec{H}, -\vec{E})$. For the experiment (B) above, one obtains then $|\Omega| = \omega_L\gamma \times [(g/2 - 1)^2 + (g'v/2)^2]^{1/2}$.

Discussions leading to this note were initiated when the two last-named authors were visiting Princeton University. They are indebted to J. R. Oppenheimer for making the facilities of the Institute for Advanced Study available to them.

*Research partly assisted by the Office of Scientific Research, Air Research and Development Command, and by the French Service des Poudres.

[1]H. A. Tolhoek and S. R. de Groot, Physica 17, 17 (1951); H. Mendlowitz and K. M. Case, Phys. Rev. 97, 33 (1955); L. M. Carrassi, Nuovo cimento 7, 524 (1958).

[2]F. Bloch, Phys. Rev. 70, 460 (1946).

[3]J. Frenkel, Z. Physik 37, 243 (1926).

[4]H. A. Kramers, Quantum Mechanics (North Holland Publishing Company, Amsterdam, 1957), p. 226 et seq. Kramers' Eq. (4) (p. 229) does not correspond to our Eq. (6). His conclusion that already classically $g = 2$ is implied for an electron, based on the relativistic equation he uses, stems from the fact that that equation corresponds in the rest-frame to $ds^0/d\tau = (ge/2m)\vec{s}\cdot\vec{E}$. Comparing with our (4), with $d\vec{v}/d\tau = (e/m)\vec{E}$, one sees that the "derived" result is, in fact, built into the theory from the start. A more general equation is mentioned by Kramers on p. 231, in fine print, and attributed to Frenkel. The inconsistencies arising in Kramers' discussion of what he calls "spin-orbit" forces [i.e., of the form $(\nabla\vec{H})\cdot\vec{u}$ in the rest-frame] are connected with the fact that neither of his equations of motion applies when the field is inhomogeneous. In that case, $du/d\tau$ is not given by (5) alone, but has to include an additional term (the covariant analog of the gradient force just mentioned) before being introduced into (6).

[5]The covariant polarization four-vector s is essentially the expectation value of the operator w used by Bargmann and Wigner [Proc. Natl. Acad. Sci. U. S. 34, 211 (1948)] to characterize representations of the inhomogeneous Lorentz group: $\langle w\cdot w\rangle = -\mathcal{S}(\mathcal{S}+1)m^2$, $\mathcal{S}$ = spin. s can be expressed in terms of the skew tensor M of Frenkel (which satisfies $M\cdot u = 0$), and vice versa: $s = M^*\cdot u$, $M^* = s\times u$, i.e., $M^{*ik} = s^i u^k - s^k u^i$. For the quantum-mechanical applications of s see, e.g., C. Bouchiat and L. Michel, Phys. Rev. 106, 170 (1955).

[6]Our notation is: $c = 1$, $\hbar = 1$ throughout; coordinate four-vector of components $x^0 = t$, x^1, x^2, x^3: $x = (x^0, \vec{x})$, $\vec{x} = \{x^\alpha\}$ ($\alpha = 1, 2, 3$); metric of signature (+ ---); τ = proper time; a dot between symbols, contraction of neighboring indices with the metric tensor, e.g., $x\cdot x = (x^0)^2 - \vec{x}^2$; skew tensor of components T^{ik} indicated as $T = (\vec{T}^t, \vec{T}^s)$, $\vec{T}^t = \{T^{0\alpha}\}$, $\vec{T}^s = \{T^{\beta\gamma}\}$; $\alpha, \beta, \gamma = 1, 2, 3$; its dual by $T^* = (\vec{T}_s, -\vec{T}_t)$.

[7]Equations (5) and (7) can be integrated explicitly by reference to four orthonormal four-vectors $e^{(i)}$ such that each of them obeys (5), and $e^{(0)} = u$.

[8]Crane, Pidd, and Louisell, Bull. Am. Phys. Soc. Ser. II, 3, 369 (1958).

[9]H. Frauenfelder et al., Phys. Rev. 106, 386 (1957).

[10]P. E. Cavanagh et al., Phil. Mag. 2, 1105 (1957).

[11]P. S. Farago, Proc. Phys. Soc. (London) 72, 891 (1958).