ONE-MESON CONTRIBUTION TO THE DEUTERON QUADRUPOLE MOMENT*

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According to Mandelstam's representation, partial-wave amplitudes for the nucleon-nucleon scattering have singularities only on the real axis of the momentum-square variable.[1] These singularities correspond to physical processes such as the deuteron bound state, one or more meson exchange, etc. The low-energy triplet s-wave amplitude is quite accurately given by the deuteron pole alone. More precisely, the position and residue of the deuteron pole determines the scattering length and the effective range and vice versa. If one believes that the small mixture of d-state in the deuteron is mainly due to the long-range part of the nuclear force, then the residue of the deuteron pole for the mixing amplitude (mixing of s and d waves in the $J=1$ scattering state) should be largely determined by the one-meson exchange contribution. When both the deuteron pole and the one-meson branch cut are included in the calculation of the mixing amplitude, the requirement that this amplitude vanishes at zero momentum then gives the residue of the deuteron pole in terms of the meson-nucleon coupling constant. This residue is simply related to the ratio of the asymptotic s- and d-wave functions of the deuteron. The result is in good agreement with those of the phenomenological deuteron wave functions which have been adjusted to the observed quadrupole moment as well as other low-energy data. A conventional direct estimation of the quadrupole moment in terms of the asymptotic s-d wave ratio gives a value within 40% of the experimental result.

In the notation of Stapp et al.,[2] the s-wave amplitude $\alpha_{0,1}$ and the mixing amplitude α^1 are given in terms of the eigenphases $\delta_{0,1}$, $\delta_{2,1}$, and the mixing parameter ϵ_1 by

$$\alpha_{0,1}=2i(\cos^2\epsilon_1 e^{i\delta_{0,1}}\sin\delta_{0,1} + \sin^2\epsilon_1 e^{i\delta_{2,1}}\sin\delta_{2,1}),$$

$$\alpha^1=i\sin 2\epsilon_1(e^{i\delta_{0,1}}\sin\delta_{0,1}-e^{i\delta_{2,1}}\sin\delta_{2,1}), \quad (1)$$

and a corresponding expression holds for the d-wave amplitude $\alpha_{2,1}$. For small ϵ_1 and $\delta_{2,1}$, Eq. (1) can be replaced by

$$\alpha_{0,1}=2ie^{i\delta_{0,1}}\sin\delta_{0,1},$$

$$\alpha^1=i\sin 2\epsilon_1 e^{i\delta_{0,1}}\sin\delta_{0,1}. \quad (2)$$

In the effective-range approximation, we have

$$\frac{\alpha_{0,1}}{2iq}=\frac{1}{q}e^{i\delta_{0,1}}\sin\delta_{0,1}=\frac{1}{q\cot\delta_{0,1}-iq}=\frac{1}{(1/a)+\frac{1}{2}rq^2-iq}, \quad (3)$$

where q is the center-of-mass momentum of either one of the nucleons, a is the scattering length, and r is the effective range. Experimental values for a and r in meson units are[3]

$$a=-3.82, \qquad r=1.20. \quad (4)$$

It is clear from Eq. (3) that $(\alpha_{0,1}/2iq)$ has poles at

$$q^2=-\left[\frac{1}{r}\pm\left(\frac{1}{r^2}+\frac{2}{ar}\right)^{1/2}\right]^2=\begin{cases}-1.79\\-0.106.\end{cases} \quad (5)$$

As anticipated, the value $q_0^2=0.106$ is almost exactly the deuteron binding energy times the nucleon mass m (q^2/m is the kinetic energy in the center-of-mass system). The pole at -1.79 can be interpreted as an interaction that gives rise to the bound state. The residue of the deuteron pole at $-q_0^2$ is

$$\Gamma=-2q_0/(1-rq_0)=-1.07. \quad (6)$$

In the effective-range theory, $-\Gamma$ is just the square of the normalization constant of the deuteron radial s-wave function. To summarize, any two of the four parameters $\{a, r, q_0^2, \Gamma\}$ determine the remaining two inasmuch as the deuteron pole dominates the low-energy behavior of the $\alpha_{0,1}$ amplitude.

Since the deuteron is known to have a small mixture of d-state, the mixing amplitude (α^1/iq) will also have a pole at $-q_0^2$. From the nonrelativistic scattering theory,[4] the residue is $(-2\Gamma\rho)$, where ρ is the ratio of the asymptotic s- and d-wave functions of the deuteron. When the one-meson exchange process is included, a branch point will occur at $q^2=-\frac{1}{4}$. If the branch cut is taken from $-\frac{1}{4}$ to $-\infty$, the discontinuity of (α^1/iq)

across the cut can be shown to be[5]

$$\lim_{\epsilon\to 0^+}\left[\left(\frac{\alpha^1}{iq}\right)_{q^2+i\epsilon}-\left(\frac{\alpha^1}{iq}\right)_{q^2-i\epsilon}\right]$$

$$=\frac{2i\pi f^2 m}{\sqrt{2}}\left(\frac{1}{q^2}+\frac{3}{4q^4}\right)\quad\text{for}\quad q^2\leqslant -\tfrac{1}{4}. \quad (7)$$

On the positive real axis of the q^2 complex plane, the phase of (α^1/iq) from Eq. (2) is given by $\delta_{0,1}$ which can be calculated from Eq. (3) using the effective-range formula. Together with the prescribed singularities on the negative real axis, the solution of (α^1/iq) can be found by the general method of Omnès.[6] However, for the present problem, it is simple enough to write down a solution that possesses all the required analytic properties. One obtains

$$(\alpha^1/iq)=\left(\frac{1}{a}+\frac{r}{2}q^2+iq\right)\left[\frac{-2\Gamma\rho}{[1/a-(r/2)q_0^2-q_0](q^2+q_0^2)}\right.$$

$$\left.+\frac{f^2m}{\sqrt{2}}\int_{-\infty}^{-1/4}d(q'^2)\,\frac{1/q'^2+3/4q'^4}{(q'^2-q^2)[1/a+(r/2)q'^2+iq']}\right]. \quad (8$$

This should be a reasonable approximation at low energy since it reproduces exactly the deuteron pole, the one-meson cut, and the low-energy phase shift. The requirement that (α^1/iq) vanishes at zero momentum then gives

$$\rho=0.029, \quad (9)$$

for $f^2=0.08$. This value of ρ is consistent with those of the phenomenological deuteron wave functions which range between 0.025 and 0.032.[7,8]

A direct estimation of the quadrupole moment in terms of ρ can also be made in the conventional way.[3] This gives a quadrupole moment of $\sim 4\times10^{-27}$ cm^2 compared with the experimental value of 2.7×10^{-27} cm^2.

A more detailed consideration of the one-meson contribution to nucleon-nucleon scattering will be published in a separate report with H. P. Noyes.

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[1]S. Mandelstam, Phys. Rev. 112, 1344 (1958).

[2]Stapp, Ypsilantis, and Metropolis, Phys. Rev. 105, 302 (1957).

[3]For example, see H. A. Bethe and P. Morrison, Elementary Nuclear Theory (John Wiley and Sons, Inc., New York, 1956).

[4]For example, see Goldberger, Nambu, and Oehme, Ann. Phys. 2, 226 (1957).

[5]Cziffra, MacGregor, Moravcsik, and Stapp, University of California Radiation Laboratory Report UCRL-8510, 1958 (unpublished).

[6]R. Omnès, Nuovo cimento 8, 316 (1958).

[7]L. Hulthén and M. Sugawara, in Handbuch der Physik (Springer-Verlag, Berlin, 1957), Vol. 39, p. 92.

[8]Previous work on relating the coupling constant with the s-d wave ratio can be found in Iwadare, Otsuki, Tamagaki, and Watari, Progr. Theor. Phys. (Kyoto) 15, 86 (1956), and 16, 455 (1956).