in the P states in spite of the similar space properties of π and K mesons.

For the $\bar{K}$ particles the situation is different since even at threshold the scattering is complicated by the production processes. But if one ignores them in the first approximation, one can get some insight into the behavior of these particles. An examination of the structure of the fourth row of (1b) immediately shows (in the effective-range approximation we are considering) that this state is certainly a candidate for resonance, for all the terms on the right-hand side are positive; that is, r_4^C is positive definite. This is an exact analog of the 33 state in pion physics. But there is one remarkable difference in our case. The absence of a strong P state for the K particles leads to an over-all positive contribution by the integrals on the right-hand sides of (1b). Since λ_2^C is the only other positive scattering length, the 03 state can equally claim to be a resonance state provided the coupling is strong. In other words, if one assumes that both the isobaric spin states in $j=3/2$ predominate, then we arrive at the conclusion that the dispersion relations are consistent with such assumption in the above sense.

We have further examined how our discussion of S waves is affected by the preceding analysis. On the basis of qualitative arguments it is concluded that the K-N scattering should be dominated by the S waves over a considerable range, and that the sign of the phase shifts at threshold should be given correctly by the Born approximation (corresponding to a repulsive interaction). For the $\bar{K}$ particles, in the absence of any experimental data in the moderately high-energy region, no definitive conclusion can really be drawn about their S-wave scattering. The results about the K particles are in agreement with experiment,[7] and lend further support to the assumption with regard to the K-meson parity.

The author would like to express his best thanks to Professor A. Salam and Dr. P. T. Matthews for suggesting the problem and for many instructive discussions.

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[1]Chew, Low, Goldberger, and Nambu, Phys. Rev. 106, 1337 (1957).

[2]Details of the present work will be submitted for publication elsewhere.

[3]That the unphysical continuum contribution may be taken to be negligible has been effectively argued by P. T. Matthews and A. Salam, Phys. Rev. 110, 569 (1958).

[4]On the basis of this type of argument one can single out, in pion-nucleon case, the 33 state as the only consistent resonance state.

[5]Provided $g_\Lambda{}^2$ is not greater than $3g_\Sigma{}^2$; otherwise, the 01 state could be a resonance state but this would be reflected only in the total K-N scattering.

[6]Similarly one can show that r_1 is negative.

[7]Reports by M. F. Kaplon and R. H. Dalitz, in 1958 Annual International Conference on High-Energy Physics at CERN, edited by B. Ferretti (CERN, Geneva, 1958), pp. 171, 187; H. C. Burrowes et al., Phys. Rev. Lett. 2, 117 (1959).

EFFECT OF A PION-PION SCATTERING RESONANCE ON NUCLEON STRUCTURE*

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The electromagnetic properties of the nucleon have recently been studied by the dispersion-relation method.[1,2] Although qualitatively successful in accounting for the isotopic vector properties of the nucleon, these treatments proved incapable of explaining simultaneously the value of the magnetic moment and the radii of the charge and moment distributions.[3] The purpose of this Letter is to show that the inclusion of a strong pion-pion interaction could explain these aspects of nucleon structure.

Let us first consider the general formulation and solution of the dispersion relations. In the notation of reference 1, we write the following representations for the form factors[4]:

$$G_i^V(s) = \frac{1}{\pi}\int_{(2m_\pi)^2}^{\infty} \frac{g_i^V(s')\,ds'}{s'-s}, \tag{1}$$

$$G_i^S(s) = \frac{1}{\pi}\int_{(3m_\pi)^2}^{\infty} \frac{g_i^S(s')\,ds'}{s'-s}, \quad i = 1 \text{ or } 2, \tag{2}$$

where $s=(p'-p)^2=(p_0'-p_0)^2-(\vec{p}\,'-\vec{p})^2$, the square of the energy-momentum transfer four-vector. The weight functions $g_i(s)$ are related to a sum over all virtual intermediate states that can be reached from a photon and lead to a nucleonantinucleon pair. For the isotopic vector functions $g_i^V(s)$, invariance considerations show that the least massive state is the two-pion state. We shall assume, as in previous treatments,[1,2] that the two-pion contribution dominates in the dispersion integrals. On the other hand, the least massive state contributing to $g_i^S(s)$ is the three-pion state. We have nothing to say here about this contribution and shall limit ourselves to the isotopic vector properties.

In the approximation stated above, $g_1^V(s)$ and $g_2^V(s)$ are proportional to the pion form factor multiplied by the appropriate projection of the amplitude for the process $\langle N\bar{N}|\pi\pi\rangle$ in the state $J=1$, $I=1$. Using the Mandelstam representation,[5] we are able to study the analytic properties of these projections, which we label $J_i(s)$. It can be shown that in the complex s-plane these functions are analytic except for branch cuts on the real axis for $s \leqslant 4m_\pi^2[1-(m_\pi^2/4M^2)]$ and $s \geqslant (2m_\pi)^2$. The right-hand branch cut, which was not considered in previous treatments, corresponds to π-π scattering. We shall show that it has an important effect on nucleon structure.

We can then write the dispersion relation

$$J_i(s) = \frac{1}{\pi}\int_{-\infty}^{a} \frac{\mathrm{Im}J_i(s')\,ds'}{s'-s-i\epsilon} + \frac{1}{\pi}\int_{(2m_\pi)^2}^{\infty} \frac{\mathrm{Im}J_i(s')\,ds'}{s'-s-i\epsilon}, \qquad (3)$$

where $a \equiv 4m_\pi^2(1-m_\pi^2/4M^2)$. Application of the unitarity condition shows that in the region $(2m_\pi)^2 \leqslant s \leqslant (4m_\pi)^2$, the phase of $J_i(s)$ is equal to the π-π scattering phase shift δ in the $J=1$, $I=1$ state. Considering only the two-pion intermediate state, we shall use this phase relation over the entire range of the right-hand integral in Eq. (3).

Because the left-hand integral is related to the pion-nucleon scattering amplitude, we shall consider it to be a known function. Equation (3) is then an integral equation, whose general solution has been found by Omnes.[6] In our case, his solution can be modified into the more tractable form

$$J_i(s) = e^{u(s)}\frac{1}{\pi}\int_{-\infty}^{a} \frac{ds'\,\mathrm{Im}J_i(s')}{s'-s-i\epsilon}\,e^{-u(s')}, \qquad (4)$$

where

$$u(s) = \frac{1}{\pi}\int_{(2m_\pi)^2}^{\infty} \frac{\delta(s')\,ds'}{s'-s-i\epsilon}. \qquad (5)$$

It can easily be seen that Eq. (4) reproduces the content of the integral equation (3); namely, $J_i(s)$ has the proper singularities, has the phase of π-π scattering on the right-hand cut, and has the correct imaginary part on the left-hand cut. If the integral defining $u(s)$ fails to converge, one can use the subtracted form

$$u_0(s) = \frac{s}{\pi}\int_{(2m_\pi)^2}^{\infty} \frac{\delta(s')\,ds'}{s'(s'-s-i\epsilon)}. \qquad (6)$$

We now require an expression for the pion form factor, $F_\pi(s)$, which satisfies the dispersion relation:

$$F_\pi(s) = 1 + \frac{s}{\pi}\int_{(2m_\pi)^2}^{\infty} ds'\,\frac{\mathrm{Im}F_\pi(s')}{s'(s'-s-i\epsilon)}. \qquad (7)$$

Unitarity allows us to conclude that the phase of $F_\pi(s)$ is the π-π scattering phase shift in the $J=1$, $I=1$ state,[7] and again we shall use this condition over the entire range of integration. In this case Eq. (4) degenerates into the solution

$$F_\pi(s) = e^{u_0(s)}. \qquad (8)$$

Combining Eqs. (4) and (8), we find

$$g_i^V(s) = |F_\pi(s)|^2\,\frac{1}{\pi}\int_{-\infty}^{a} \frac{ds'\,\mathrm{Im}J_i(s')}{(s'-s-i\epsilon)F_\pi(s')}. \qquad (9)$$

Equation (9) reveals the important fact that, because of the phase conditions imposed by unitarity, it is the absolute value of the pion electromagnetic form factor which appears in the weight functions $g_i^V(s)$. Thus the well-known condition that $g_i(s)$ be real is satisfied.

Using Eqs. (9) and (1), let us now investigate the effect of π-π scattering on the nucleon structure. It has been shown by Drell[3] that in order to obtain agreement with the nucleon magnetic moment and radii, an enhancement of $g_i^V(s)$ by a factor of the order of five is required for $s < M^2$. From Eq. (9) it is apparent that a suitable peak in the pion form factor would produce this enhancement. We shall now show that a π-π resonance would result in such a peak.

An investigation of π-π scattering now in progress by Chew and Mandelstam has shown that the singularities of the partial-wave amplitude in the $J=1$, $I=1$ state are confined to branch cuts along the real axis in the range $s \leq 0$, $4m_\pi^2 \leq s$.[8] In the physical region, the effect of the left-hand singularities can be estimated by replacing the branch cut by a pole of appropriate position and residue. This approximation seems reasonable because for nucleon-nucleon scattering it leads to well-known effective-range formulas. Making this approximation, one finds the following solution for the $J=1$ state, for $\nu > 0$:

$$f_{\pi\pi}(s) = \left(\frac{\nu+1}{\nu^3}\right)^{1/2} e^{i\delta}\sin\delta = \frac{\Gamma}{\nu_r - \nu[1-\Gamma\alpha(\nu)] - i\Gamma[\nu^3/(\nu+1)]^{1/2}}, \quad (10)$$

where

$$\nu = \tfrac{1}{4}s - m_\pi^2,$$

$$\alpha(\nu) = (2/\pi)[\nu/(\nu+1)]^{1/2}\ln[\nu^{1/2} + (\nu+1)^{1/2}].$$

A suitably continued form holds for $\nu < 0$. The constants Γ and ν_r are determined by the position and residue of the pole. By examination of the structure of the π-π equations, Chew and Mandelstam have found that the sign of the residue must be positive, corresponding to an attractive force and raising the possibility of a resonance. Further theoretical information about the equivalent pole must await numerical solution of the very complicated π-π equations. We shall now show, however, that if the constants ν_r and Γ are properly chosen, agreement with the nucleon-structure data may be achieved.

A properly normalized solution for $F_\pi(s)$ is

$$F_\pi(s) = f_{\pi\pi}(s)(s+s_0)/[s_0 f_{\pi\pi}(0)], \quad (11)$$

where s_0 is the position of the equivalent pole. The justification of this solution is, again, that it has the correct singularities and phase. Equation (11) clearly shows that a resonance in $f_{\pi\pi}$ will be reflected in the form factor F_π. The ob-

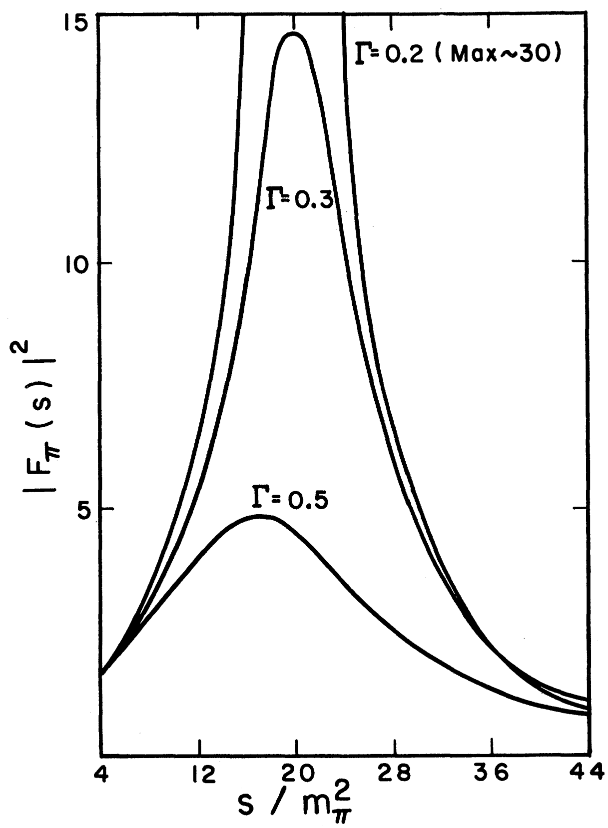

FIG. 1. The square of the magnitude of the pion form factor for $s \geq 4\mu^2$, for three values of the width Γ.

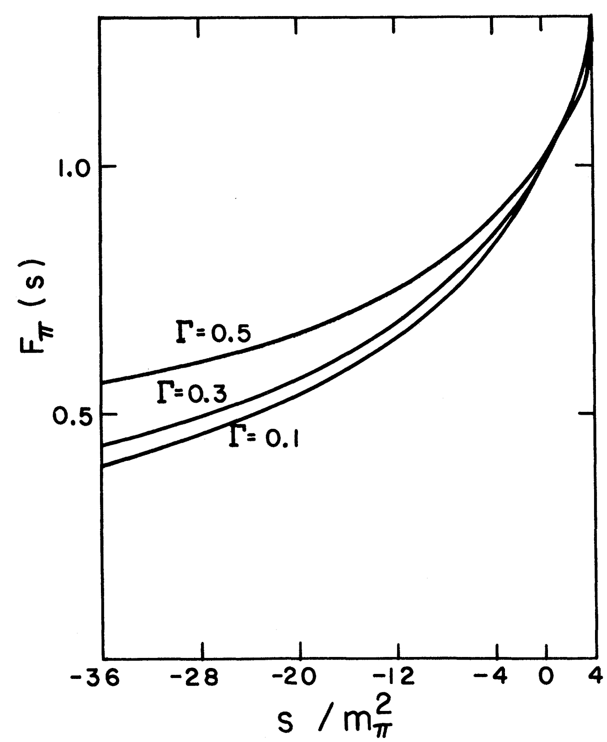

FIG. 2. The pion form factor in its physical region, for three values of the width Γ.

served values of the charge and magnetic-moment radii indicate that this resonance should occur at $\nu_\gamma \sim 3.5 m_\pi^2$ (square of the pion momentum in the π-π barycentric system). In Fig. 1 the function $|F_\pi(s)|^2$ is plotted for this value of ν_γ and several values of the width Γ. In Fig. 2 the pion form factor is plotted for $s < a$. Since it is less than one over most of this region, its appearance in the denominator of the integral of Eq. (9) will produce an additional enhancement.

In conclusion, our Eq. (9) for the weight functions together with the approximation given in Eq. (10) for the pion-pion scattering amplitude suggests that a π-π resonance of suitable position and width could lead to agreement between dispersion theory and many aspects of nucleon electromagnetic structure. Detailed calculations are in progress.

We are indebted to Professor Geoffrey F. Chew for his advice throughout this work, and for advance communication of some of the results on pion-pion scattering. We also acknowledge the help of James S. Ball and Peter Cziffra in obtaining Eq. (4).

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[1]Chew, Karplus, Gasiorowicz, and Zachariasen, Phys. Rev. 110, 265 (1958).

[2]Federbush, Goldberger, and Treiman, Phys. Rev. 112, 642 (1958).

[3]S. D. Drell, 1958 Annual International Conference on High-Energy Physics at CERN, edited by B. Ferretti (CERN, Geneva, 1958).

[4]If the integral in Eq. (1) fails to converge, one can use the usual subtracted form, as discussed in reference 1.

[5]S. Mandelstam, Phys. Rev. 112, 1344 (1958).

[6]R. Omnes, Nuovo cimento 8, 316 (1958).

[7]See, for example, Appendix II of Fubini, Nambu, and Wataghin, Phys. Rev. 111, 329 (1958).

[8]G. F. Chew, Lawrence Radiation Laboratory (private communication, 1959).

ENERGY OF THE GRAVITATIONAL FIELD*

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Einstein's equations for the gravitational field are valid for any system of coordinates and make it difficult for one to distinguish physical effects from effects of the curvature of the coordinate system. In consequence there is no obvious definition for the energy of the gravitational field. The usual definition, in terms of a component t_0^0 of the stress pseudotensor, makes the energy depend very much on the system of coordinates and is thus not satisfactory.

For physical problems one can restrict the gravitational field to be weak, of the order of γ, the gravitational constant. One can then use a system of coordinates for which the $g_{\mu\nu}$ differ from their values in special relativity by quantities of the order γ. Even if one restricts oneself to such coordinate systems (and renounces, for example, the use of polar coordinates), one can still make arbitrary changes of order γ in the coordinate system and such changes produce changes in the energy of the same order as the energy itself, so the difficulty persists.

In discussing the question people have usually lost sight of the primary requirement for the energy, that it shall be a useful integral of the equations of motion. The development of the Hamiltonian form of gravitational theory[1] enables one to make a new attack on the problem, taking this utility requirement into account.

In the Hamiltonian form one deals with the state at a certain time x^0, which state is described by dynamical variables for all values of x^1, x^2, x^3 for this one value of x^0. It is found that the only variables needed to describe the gravitational field are the six g_{rs} $(r, s = 1, 2, 3)$ and their conjugate momenta p^{rs}. The four $g_{\mu 0}$ do not enter into the description of the state at a certain time. They are needed only to provide a connection between the state at one time and the state at a neighboring time.

We are thus led to the condition that the energy at a certain time should involve only the g_{rs}, p^{rs}, and the nongravitational variables and should not depend on the $g_{\mu 0}$. The usual definition in terms of t_0^0 does not satisfy this condition, nor does the definition recently proposed by Møller.[2]