[3]K. G. Malmfors, Arkiv Fysik 1, 569 (1950).
[4]E. P. Gross, Phys. Rev. 82, 232 (1951).
[5]H. K. Sen, Phys. Rev. 88, 816 (1952).
[6]Alfvén, Lindberg, Malmfors, Wallmark, and Aström, Kgl. Tek. Högskol. Handl. No. 22 (1948).
[7]Equation (46) of reference 1.

DEVELOPMENT OF HYDROMAGNETIC SHOCKS FROM LARGE-AMPLITUDE ALFVEN WAVES

David Montgomery*
Palmer Physical Laboratory,
Princeton, New Jersey
(Received December 19, 1958)

Several authors, notably de Hoffmann and Teller,[1] have treated the conditions which obtain across a fully developed shock front in an ionized gas in the absence of external fields. Petschek[2] has derived the equations governing the growth in time of such a shock from a magnetosonic pulse of large amplitude. Ferraro[3] has discussed large-amplitude, circularly-polarized Alfvén waves and has found that such waves, if thermal motions are negligible and if there are initially no forces in the direction of propagation, will propagate undistorted in time. However, it is shown here that plane-polarized Alfvén waves of large amplitude develop rapidly into hydromagnetic shocks.

Consider the motion of an ionized gas consisting of electrons of mass m_- and positive ions of mass m_+, each of equilibrium density n_0 cm^{-3}, infinite in all directions, and immersed in a constant external magnetic field B_0 directed along the x-axis. Assume that (1) the gas remains electrically neutral to a high degree throughout the motion; (2) $n_0kT/B_0^2 << 1$, where T is the maximum temperature at any point in the gas; (3) the mean free path for collisions is >> all the characteristic lengths of the motion; (4) the displacement current is always << the conduction current; and (5) $B_0^2/4\pi n_0(m_+ + m_-) << c^2$. The circumstances under which (1)-(5) apply are well known.[4]

Consider the following progressive pulse:

$$\vec{B} = B_0\hat{i} + B_y(x,t)\hat{j},$$
$$\vec{E} = E_z(x,t)\hat{k},$$
$$\vec{J} = J_z(x,t)\hat{k}. \qquad (1)$$

Note that the average electron and ion velocities in the x and y directions, V_x and V_y, are equal. The electric current is $\vec{J}$.

The subsequent motion will then be governed by the Boltzmann-Vlasov equations:

$$\frac{\partial f_-}{\partial t} + \vec{v}\cdot\nabla f_- - \frac{e}{m_-}(\vec{E} + \vec{v}\times\vec{B})\cdot\nabla_v f_- = 0, \qquad (2a)$$

$$\frac{\partial f_+}{\partial t} + \vec{v}\cdot\nabla f_+ + \frac{e}{m_+}(\vec{E} + \vec{v}\times\vec{B})\cdot\nabla_v f_+ = 0, \qquad (2b)$$

and by the Maxwell equations:

$$\nabla\cdot\vec{E} = 0, \qquad (3a)$$
$$\nabla\cdot\vec{B} = 0, \qquad (3b)$$
$$\nabla\times\vec{E} = -\partial\vec{B}/\partial t, \qquad (3c)$$
$$\nabla\times\vec{B} = 4\pi\vec{J}, \qquad (3d)$$

where

$$\vec{J} \equiv e\int f_+\vec{v}d^3v - e\int f_-\vec{v}d^3v, \qquad (4a)$$
$$\vec{V} \equiv [m_+\int f_+\vec{v}d^3v + m_-\int f_-\vec{v}d^3v]/(m_+ + m_-). \qquad (4b)$$

Note that (3a) and (3b) are automatically satisfied. Further, assume that $\partial f_\pm/\partial y = \partial f_\pm/\partial z = 0$.

Take zeroth-order moments of Eqs. (2) and add:

$$d\rho/dt + \rho\,\partial V_x/\partial x = 0, \qquad (5)$$

where ρ is the mass density, and where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + V_x\frac{\partial}{\partial x}. \qquad (6)$$

Taking first-order moments gives, using assumption (2) and Eq. (3d),

$$\rho\frac{d\vec{V}}{dt} = \vec{J}\times\vec{B} = \frac{(\nabla\times\vec{B})\times\vec{B}}{4\pi}, \qquad (7)$$

the x-component of which is

$$\rho\frac{dV_x}{dt} + \frac{\partial}{\partial x}\left(\frac{B_0^2 + B_y^2(x,t)}{8\pi}\right) = 0. \qquad (8)$$

Taking the first moment of (2a) alone and passing to the limit $m_-/e\to 0$ yields[5]

$$E_z + V_xB_y - V_yB_0 = 0, \quad V_{z-} = 0. \qquad (9)$$

Equations (3c), (5), (8), and (9) can be combined to give (at least through third order in B_y/B_0)

$$(B_0^2 + B_y^2)^{1/2}/\rho = \text{const.} \qquad (10)$$

It will now be apparent that Eqs. (5), (8), and (10) are nothing more than the equations for a nonlinear sound wave from ordinary gas dynamics, with the replacement of $(B_0^2 + B_y^2)/8\pi$ for the pressure, a frequent result in plasma dynamics. Equation (10) is the analog of the adiabatic relation $P/\rho^\gamma = \text{const}$, with $\gamma = 2$. The ordinary gas-dynamical theory can be applied; in particular, the distortion of the wave profile in a plot of $B_y^2(x,t)$ vs x can be followed until a vertical tangent forms (i.e., a shock front) by the usual geometrical construction. The local velocity of the wave motion is a,

$$a^2 \equiv \frac{d}{d\rho}\left(\frac{B_0^2 + B_y^2}{8\pi}\right) = \frac{B_0^2 + B_y^2}{4\pi\rho}, \qquad (11)$$

which, of course, approaches the Alfvén velocity as $B_y/B_0 \to 0$. Points of higher B_y^2 will gain on points of lower, and the pulse will steepen to a discontinuity. One point on a plot of B_y^2 as a function of x becomes vertical after a time t_s, which is readily shown to be[6]

$$t_s \approx \frac{2B_0(1 + B_y^2/B_0^2)^{3/4}(4\pi\rho_0)^{1/2}}{B_y |dB_y/dx|_{\max}} \qquad (12)$$

All variable quantities are evaluated at the point of maximum $|dB_y/dx|$ at $t = 0$. The shock time t_s is of the order of one period if B_y/B_0 is of the order of 1 and B_y is approximately sinusoidal at $t = 0$. Since the system of Eqs. (2)-(3) conserves entropy,[7] the discontinuity differs from acoustical shocks in that it is nondissipative; this is because the random or thermal effects have been neglected by assumption (2).

It is not clear what process ultimately limits the thickness of the shock; an eventual charge separation seems more likely than the dissipation of energy by viscosity of ordinary acoustical shocks.[2] It would also be of interest to calculate the stability of the motion.

---

*National Science Foundation Predoctoral Fellow.

[1]F. de Hoffmann and E. Teller, Phys. Rev. 80, 692 (1950).

[2]H. E. Petschek, Proceedings of the Third Symposium on Cosmical Gas Dynamics [Revs. Modern Phys. 30, 966 (1958)].

[3]V. C. A. Ferraro, Proc. Roy. Soc. (London) A233, 310 (1955).

[4]See, for instance, Bernstein, Frieman, Kruskal, and Kulsrud, Proc. Roy Soc. (London) A244, 17 (1958).

[5]Sometimes loosely called the "infinite conductivity" assumption, this is mathematically equivalent to making the electron Larmor radius small compared to the distances over which the fields vary appreciably. See Chew, Goldberger, and Low, Proc. Roy. Soc. (London) A236, 11 (1956).

[6]The time of steepening is worked out in detail for an acoustical wave by G. I. Taylor and J. W. Maccoll, in Aerodynamic Theory, edited by W. F. Durand (California Institute of Technology, Pasadena, 1943), Vol. 3, pp. 210-217. The calculation leading to (12) is similar.

[7]Ira. B. Bernstein, Phys. Rev. 109, 10 (1957).

EXCITATION OF OSCILLATIONS IN A PLASMA LAYER

M. Sumi
Electrical Communication Laboratory,
Tokyo, Japan
(Received December 15, 1958)

Recently the author developed a linearized theory of excited plasma waves,[1] in which electron plasma oscillations excited by an injected beam into a uniform plasma were discussed in detail. The dispersion relation in this case was obtained as follows:

$$(\omega_0/\omega_p)^2 = 1 + (3/2\beta^2) + (15/4\beta^4) + \dots + \sigma\beta^2/(\xi - \beta)^2, \qquad (1)$$

where $\beta = \omega/ku_b$, $\xi = u_b/u_T$, $\omega_p^2 = n_pe^2/m\epsilon_0$, and $\sigma = n_b/n_p$. u_b and n_b denote uniform velocity and electron density in the beam respectively, while $u_T = (2\kappa T/m)^{1/2}$ and n_p are thermal velocity and electron density in the plasma, respectively. The relation (1) was derived without any ambiguity by using an approximation for $|\beta| \gg 1$. Although similar dispersion relations have been derived and discussed to some extent by several authors,[2-5] more exact and detailed characteristics of excited waves were needed for our purpose. Equation (1) was solved for an excited wave, where $\text{Im}(\omega) = \gamma > 0$, the complex frequency being written as $\omega = \omega_0 + i\gamma$. Each wave component builds up from a ground level of initial disturbance proportionally to $\exp(\gamma t)\exp(kz - \omega_0 t)$. Except for very small values of σ, Eq. (1) must be solved numerically. The typical case with $\sigma = 0.1$ and $(u_b/u_T)^2 \to \infty$ is shown in Fig. 1, where the frequency ω_0 and the wave growth rate γ, both normalized by ω_p, and the ratio of plasma wave velocity $u_w = \omega_0/k$ to the beam velocity u_b are plotted against the wave number k. These characteristics are found to be not very sensitive to finite values of $(u_b/u_T)^2$.

The theory is applied to the excitation of a standing wave in a uniform plasma layer with a thickness D. The wavelength λ of the standing wave is determined by the relation $D = n(\lambda/2)$,