not succeed. I came to the conclusion that Rickayzen's method, in the Meissner effect problem, does not provide a proper starting point for subsequent low-order approximations, and that, at any rate, his power expansions in g cannot be trusted. If high-order terms contribute essentially to $\vec{j}_{\text{long}}$, they can certainly also affect $\vec{j}_{\text{trans}}$. The "leading term" may well be equally misleading in both cases.

In view of these doubts, it seemed a decisive advantage first to transform the Hamiltonian into a manifestly gauge-invariant form [reference 2, Eq. (2) with (1) and (17)]. In this new representation, the current operator $\vec{j}$ is again obtained as a power series in g [Eq. (10)], but now $\vec{j}_{\text{long}}$ vanishes automatically, to all orders in g. This does not prove, of course, that the power series for $\vec{j}_{\text{trans}}$ converges rapidly. But Pines' and Schrieffer's criticism of this expansion is groundless and futile because it is based on a comparison with a low-order approximation that is in essence the same as Rickayzen's.[7]

[1]D. Pines and J. R. Schrieffer, Phys. Rev. Lett. 1, 407 (1958). Dr. Schrieffer kindly gave me a copy, and we discussed the subject.

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[4]J. Bardeen, Nuovo cimento 5, 1765 (1957); P. W. Anderson, Phys. Rev. 110, 827 (1958); D. Pines and J. R. Schrieffer, Nuovo cimento (to be published).

[5]E. N. Adams, Phys. Rev. 98, 1130 (1955).

[6]Corresponding to isolated roots of a secular equation. Compare Bogoliubov, Tolmachov, and Shirkov, "A new method in the theory of superconductivity," Dubna, June, 1958, sections 3.2-4; P. W. Anderson, Phys. Rev. (to be published).

[7]To realize this, one need only follow the argumentation in reference 1 (on p. 408). Note the replacement of Ψ_α and Ψ_β by eigenfunctions of a reduced Hamiltonian, and compare with reference 3.

UNSTABLE PLASMA OSCILLATIONS IN A MAGNETIC FIELD

E. G. Harris*
Oak Ridge National Laboratory, †
Oak Ridge, Tennessee
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This work treats the small amplitude oscillations of a fully ionized quasi-neutral plasma in a uniform time-independent externally produced magnetic field. Motions of the ions and perturbations of the magnetic field are neglected. The distribution function $f(\vec{r}, \vec{v}, t)$ for the electrons satisfies

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{r}} - \frac{e}{m}\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)\cdot\frac{\partial f}{\partial \vec{v}} = 0, \tag{1}$$

and the electric field which appears in Eq. (1) satisfies

$$\nabla\cdot\vec{E} = -\nabla^2\phi = -4\pi e\int f d^3v. \tag{2}$$

The distribution function is assumed to depart only slightly from the zeroth order distribution, and the spatial dependence of the perturbation of the distribution is assumed to be given by the factor $\exp(i\vec{k}\cdot\vec{r})$. Equations (1) and (2) are linearized and then solved by taking the Laplace transform and following the procedure of Bernstein.[1] It is found that the Laplace transform of the potential is given by

$$\phi(s) = \left[-\frac{4\pi e}{k^2\omega_c}\int g(\vec{v}, \vec{k}, s)d^3v\right] / [1 - Y(s)]. \tag{3}$$

In Eq. (3), s is the Laplace transform parameter, $\omega_c = eB/mc$ is the cyclotron frequency and $g(\vec{v}, \vec{k}, s)$ is a function related to the initial value of the perturbation of the distribution function. $Y(s)$ is given by

$$Y(s) = 2\pi i\frac{\omega_p^2}{k^2}\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty} dv_z\int_0^\infty v_\perp dv_\perp \times\left\{\frac{\omega_c}{v_\perp}\frac{\partial F}{\partial v_\perp}\frac{nJ_n^2(k_\perp v_\perp/\omega_c)}{(s+ik_zv_z+in\omega_c)} + k_z\frac{\partial F}{\partial v_z}\frac{J_n^2(k_\perp v_\perp/\omega_c)}{(s+ik_zv_z+in\omega_c)}\right\}. \tag{4}$$

In Eq. (4), $\omega_p = (4\pi Ne^2/m)^{1/2}$ is the plasma frequency, k_z is the component of $\vec{k}$ along $\vec{B}$, and $k_\perp$ is the perpendicular component. Similarly, v_z is the component of $\vec{v}$ along $\vec{B}$ and $v_\perp$ is the perpendicular component. J_n is the Bessel function of order n. $F(v_z, v_\perp)$ is the zeroth order distribution. It is normalized so that its integral over all of velocity space is unity.

We are particularly interested in zeroth order distributions which cause the denominator of Eq. (3) to vanish for some values of s which have positive real parts. That is,

$$Y(s) = 1 \quad \text{for } \mathrm{Re}(s)>0. \tag{5}$$

If Eq. (5) is satisfied then there will exist plasma oscillations whose amplitudes increase exponentially. We shall exhibit two distributions for which this is true. First, we consider

$$F(v_z, v_\perp) = \frac{1}{2\pi}\,\delta(v_z)\,\frac{\delta(v_\perp - V)}{v_\perp}. \tag{6}$$

Equation (4) becomes

$$Y(s) = -\left(\frac{\omega_p}{\omega_c}\right)^2 \sum_{n=-\infty}^{+\infty} \left\{ i\left(\frac{k_\perp}{k}\right)^2 \frac{n}{(s/\omega_c + in)}\left[\frac{1}{b}\frac{d}{db}J_n{}^2(b)\right] + \left(\frac{k_z}{k}\right)^2 \frac{1}{(s/\omega_c + in)^2}\left[J_n{}^2(b)\right]\right\}, \tag{7}$$

where $b = k_\perp V/\omega_c$.

In order to determine whether unstable oscillations exist we use the Nyquist criterion.[2] That is, we note that $Y(s)$ defines a mapping of the right half of the s-plane onto a region of the $Y(s)$ plane, and the boundary of this region is the curve $Y = Y(i\omega)$, $-\infty < \omega + \infty$. Equation (5) will be satisfied if the curve $Y = Y(i\omega)$ encloses the point $+1$. If we plot the Nyquist diagram of Eq. (7) we find that for a suitable choice of b, $k_\perp$, h_z, and (ω_p/ω_c) unstable oscillations exist. If $k_z = 0$, unstable oscillations exist only for $b > 1.84$ [the first maximum of $J_1(b)$].

The distribution given by Eq. (6) was first studied by Malmfors[3] who concluded that instabilities existed. However, an error in his work was found by Gross.[4] Malmfors and Gross considered only the case $k_z = 0$. Gross showed that there was no instability for $b \ll 1$ and conjectured that there was no instability for any b. Sen[5] was able to show numerically that there were instabilities for large b. Sen also considered only the case $k_z = 0$. Our analysis indicates that the greatest instability occurs when neither k_z nor $k_\perp$ are zero. Instabilities of this sort may, as Malmfors suggests, account for the noise in trochotrons reported by Alfven et al.[6]

We have also considered the distribution

$$F(v_z, v_\perp) = \frac{\alpha_z}{\pi^2 \alpha_\perp{}^2}\,\frac{\exp(-v_\perp{}^2/\alpha_\perp{}^2)}{v_z{}^2 + \alpha_z{}^2}. \tag{8}$$

This distribution was chosen because it resembles the Maxwell-Boltzmann function and allows the integrals in Eq. (4) to be evaluated in terms of known functions. The isotropy of the distribution can be varied by changing $\alpha_\perp$ and α_z.

Substituting Eq. (3) into Eq. (4) gives

$$Y(s) = -\left(\frac{\omega_p}{\omega_c}\right)^2 \sum_{n=-\infty}^{+\infty} e^{-\lambda} I_n(\lambda) \times \left\{ \frac{i}{\lambda}\left(\frac{k_\perp}{k}\right)^2 \frac{n}{[(s + |k_z \alpha_z|)/\omega_c + in]} + \left(\frac{k_z}{k}\right)^2 \frac{1}{[(s + |k_z \alpha_z|)/\omega_c + in]^2}\right\}, \tag{9}$$

where $\lambda = \alpha_\perp{}^2 k_\perp{}^2/2\omega_c{}^2$ and $I_n(\lambda) = I_{-n}(\lambda)$ is the modified Bessel function of the first kind. When $k_z = 0$ Eq. (9) may be shown to agree with a result of Bernstein.[7] There is neither instability nor damping. It may also be shown that there can be no instability for $k_\perp = 0$ although there will be Landau damping unless $\alpha_z = 0$. Instability can only occur when neither $k_\perp$ nor k_z is zero.

We have considered the intermediate case $k_\perp = k_z$ with $\lambda = 0.5$ and $\alpha_z = 0$. A plot of the Nyquist diagram showed that there were unstable oscillations for $\omega_p/\omega_c > 1.1$.

It is apparent from the structure of Eq. (9) that a nonzero value of α_z will decrease the instability or increase the damping of the oscillations.

The instabilities discussed here may have serious consequences for attempts to achieve controlled thermonuclear reactions. The instabilities are due to the anisotropy of the velocity distributions. In most thermonuclear devices anisotropies naturally arise. In high-energy injection devices such as the Oak Ridge DCX and the Russian OGRA the average particle velocities are greater perpendicular to the field than along the field. The same thing is true to a somewhat smaller extent in any machine that relies on magnetic mirrors for containment. Machines of the Stellarator type may produce anisotropies by magnetic pumping. A further study of these instabilities is in progress.

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*Consultant from University of Tennessee, Knoxville, Tennessee.

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DEVELOPMENT OF HYDROMAGNETIC SHOCKS FROM LARGE-AMPLITUDE ALFVEN WAVES

David Montgomery*
Palmer Physical Laboratory,
Princeton, New Jersey
(Received December 19, 1958)

Several authors, notably de Hoffmann and Teller,[1] have treated the conditions which obtain across a fully developed shock front in an ionized gas in the absence of external fields. Petschek[2] has derived the equations governing the growth in time of such a shock from a magnetosonic pulse of large amplitude. Ferraro[3] has discussed large-amplitude, circularly-polarized Alfvén waves and has found that such waves, if thermal motions are negligible and if there are initially no forces in the direction of propagation, will propagate undistorted in time. However, it is shown here that plane-polarized Alfvén waves of large amplitude develop rapidly into hydromagnetic shocks.

Consider the motion of an ionized gas consisting of electrons of mass m_- and positive ions of mass m_+, each of equilibrium density n_0 cm^{-3}, infinite in all directions, and immersed in a constant external magnetic field B_0 directed along the x-axis. Assume that (1) the gas remains electrically neutral to a high degree throughout the motion; (2) $n_0kT/B_0^2 \ll 1$, where T is the maximum temperature at any point in the gas; (3) the mean free path for collisions is $\gg$ all the characteristic lengths of the motion; (4) the displacement current is always $\ll$ the conduction current; and (5) $B_0^2/4\pi n_0(m_+ + m_-) \ll c^2$. The circumstances under which (1)-(5) apply are well known.[4]

Consider the following progressive pulse:

$$\vec{B} = B_0\hat{i} + B_y(x,t)\hat{j},$$
$$\vec{E} = E_z(x,t)\hat{k},$$
$$\vec{J} = J_z(x,t)\hat{k}. \qquad (1)$$

Note that the average electron and ion velocities in the x and y directions, V_x and V_y, are equal. The electric current is $\vec{J}$.

The subsequent motion will then be governed by the Boltzmann-Vlasov equations:

$$\frac{\partial f_-}{\partial t} + \vec{v}\cdot\nabla f_- - \frac{e}{m_-}(\vec{E} + \vec{v}\times\vec{B}) \cdot \nabla_v f_- = 0, \qquad (2a)$$

$$\frac{\partial f_+}{\partial t} + \vec{v}\cdot\nabla f_+ + \frac{e}{m_+}(\vec{E} + \vec{v}\times\vec{B}) \cdot \nabla_v f_+ = 0, \qquad (2b)$$

and by the Maxwell equations:

$$\nabla\cdot\vec{E} = 0, \qquad (3a)$$
$$\nabla\cdot\vec{B} = 0, \qquad (3b)$$
$$\nabla\times\vec{E} = -\partial\vec{B}/\partial t, \qquad (3c)$$
$$\nabla\times\vec{B} = 4\pi\vec{J}, \qquad (3d)$$

where

$$\vec{J} \equiv e\int f_+\vec{v}d^3v - e\int f_-\vec{v}d^3v, \qquad (4a)$$
$$\vec{V} \equiv [m_+\int f_+\vec{v}d^3v + m_-\int f_-\vec{v}d^3v]/(m_+ + m_-). \qquad (4b)$$

Note that (3a) and (3b) are automatically satisfied. Further, assume that $\partial f_\pm/\partial y = \partial f_\pm/\partial z = 0$.

Take zeroth-order moments of Eqs. (2) and add:

$$d\rho/dt + \rho\partial V_x/\partial x = 0, \qquad (5)$$

where ρ is the mass density, and where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + V_x\frac{\partial}{\partial x}. \qquad (6)$$

Taking first-order moments gives, using assumption (2) and Eq. (3d),

$$\rho\frac{d\vec{V}}{dt} = \vec{J}\times\vec{B} = \frac{(\nabla\times\vec{B})\times\vec{B}}{4\pi}, \qquad (7)$$

the x-component of which is

$$\rho\frac{dV_x}{dt} + \frac{\partial}{\partial x}\left(\frac{B_0^2 + B_y^2(x,t)}{8\pi}\right) = 0. \qquad (8)$$

Taking the first moment of (2a) alone and passing to the limit $m_-/e \to 0$ yields[5]

$$E_z + V_xB_y - V_yB_0 = 0, \quad V_{z_-} = 0. \qquad (9)$$

Equations (3c), (5), (8), and (9) can be combined to give (at least through third order in B_y/B_0)

$$(B_0^2 + B_y^2)^{1/2}/\rho = \text{const.} \qquad (10)$$

It will now be apparent that Eqs. (5), (8), and (10) are nothing more than the equations for a nonlinear sound wave from ordinary gas dynamics, with the replacement of $(B_0^2 + B_y^2)/8\pi$ for the pressure, a frequent result in plasma dynam-