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DETERMINATION OF THE SUPERCONDUCTING SKIN DEPTH FROM THE ENERGY GAP AND SUM RULE*

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It has previously been pointed out[1] that the oscillator-strength sum rule provides a useful relation between the energy gap in a superconductor and the strength of the supercurrent which flows in response to an applied low-frequency field. A direct consequence is the effect of impurities in increasing the penetration depth.[2] The purpose of the present note is to extend these arguments to the case of an ideal pure bulk superconductor. In this way we are led to a nonlocal relation of current to field which is independent of any detailed model. This relation is equivalent to that proposed by Pippard[3] on a relatively heuristic basis to explain his experimental results, and by Bardeen, Cooper, and Schrieffer[4] (BCS) on the basis of their detailed microscopic model of superconductivity. Our result is an explicit confirmation of a suggestion made a few years ago by Bardeen,[5] that the electromagnetic properties of a superconductor are immediate consequences of its energy gap. Our proof is, however, more straightforward than Bardeen's, in that it makes use of the Kramers-Kronig relations and the sum rule to introduce the effect of the energy gap. Thus, only matrix elements for true physically observable absorptive processes are involved, while Bardeen's calculation dealt with matrix elements for virtual excitations concerning which it was necessary to make conjectures. The present approach is restricted to oscillating fields and consequently does not actually describe the Meissner effect (expulsion of a dc magnetic field). But the expulsion of an ac field is accounted for, and since the penetration depth which we determine is frequency independent, it follows that it must be equal to that for the dc limit of zero frequency. (See also our concluding remarks.)

To provide a general framework for the analysis, we Fourier-analyze all space- and time-dependent quantities. The coefficients of the plane wave $\exp[i(\vec{q}\cdot\vec{x}-\omega t)]$ in the current density and electric field will be denoted by $\vec{J}(\omega,\vec{q})$ and $\vec{E}(\omega,\vec{q})$, respectively. The ratio of current to field defines a complex conductivity, $\sigma(\omega,q) = \sigma_1(\omega,q)+i\sigma_2(\omega,q)$, which is a function of the wave number q as well as the angular frequency ω. What we seek to establish for the superconducting state is a certain dependence of $\sigma_2(\omega,q)$ on q, for small values of ω. If $\sigma_2(\omega,q)$ were independent of q, then the inverse Fourier transform would yield a delta function in configuration space, corresponding to a strictly local relationship between current density and applied field. On the other hand, if $\sigma_2(\omega,q)$ were to diminish as q increased, then the supercurrent which flowed in response to a spatially varying field would not have its full London value, and the relationship of current to field would be of the same type as the nonlocal one expressed by Pippard's theory. We shall see by arguments based upon quite general principles that the latter situation is the case.

The concept of a wave number as well as frequency-dependent conductivity is equivalent to a similar concept for the dielectric constant which has been explored thoroughly by Lindhard[6] for the normal state of the metal. Assuming that q is much less than the Fermi wave vector, and making a sign correction in his Eqs. (3.15) and (3.17), we obtain $\sigma_1 = 0$ for $\omega > v_0 q$, and

$$\sigma_1(\omega, q) = \frac{3\pi}{4}\frac{ne^2}{mv_0 q}\left(1 - \frac{\omega^2}{v_0^2 q^2}\right), \qquad (1)$$

for $\omega < v_0 q$, where n, e, and m are the density, charge, and mass of the free electrons. An alternative derivation of (1) is to take the limit as $l \to \infty$ of the Fourier transform of the integral expression given by Chambers[7] for the retarded nonlocal relationship of current density to electric field. Explicit integration of (1) yields

$$\int_0^\infty \sigma_1(\omega, q)\, d\omega = \pi ne^2/2m, \qquad (2)$$

independent of q. The sum rule[8] states that (2) must hold regardless of the details of the system. (We neglect throughout the small contribution to the conductivity from the motion of the ions.)

If now, for some reason the conductivity is modified so that the area under the $\sigma_1(\omega, q)$ vs ω curve is apparently diminished by the amount A, the sum rule (2) requires that the "lost" area be compensated by a delta function of strength A at $\omega = 0$. This in turn produces a contribution to the imaginary part of the conductivity, according to the Kramers-Kronig causality relations, of amount

$$\sigma_2^{\,A}(\omega, q) = 2A/\pi\omega. \qquad (3)$$

It is known empirically[9] that the effect of the transition to the superconducting state is to remove an area A which may conveniently be expressed as $\tilde{\omega}_g \sigma_1(0, q)$, the product of an effective gap frequency $\tilde{\omega}_g$ and the value of $\sigma_1(\omega, q)$ at zero frequency. This is true at least when $\sigma_1(\omega, q)$ is nearly constant[10] for $\omega \lesssim \tilde{\omega}_g$, as it is for $v_0 q \gg \tilde{\omega}_g$. The effective gap width $\hbar\tilde{\omega}_g$ defined in this way is about twice the threshold energy $\hbar\omega_g$ or about $8kT_c$. Replacing A in (3) by $\tilde{\omega}_g \sigma_1(0, q)$ and using (1), we find an imaginary conductivity equal to $(ne^2/m\omega)F(q)$, where $F(q)$ measures the q-dependent reduction of the London conductivity given by the first factor. For $q \gg \tilde{\omega}_g/v_0$, we find explicitly

$$F(q) = 3\,\tilde{\omega}_g/2v_0 q. \qquad (4)$$

For $q = 0$, we expect $F(0) = 1$, since for $q \approx 0$, the entire area under $\sigma_1(\omega, q)$ as given by (1) falls well below ω_g, yielding the full sum rule value for A, and hence the full London conductivity.[11] The difference in the two cases is illustrated in Fig. 1.

The penetration depth can now be calculated from the familiar formula[4] for diffuse reflection:

$$\lambda = \pi \Big/ \int_0^\infty \ln[1 + F(q)/q^2\lambda_L^{\,2}]\, dq, \qquad (5)$$

where λ_L is the London penetration depth $(mc^2/4\pi e^2)^{1/2}$. Assuming (4) to hold for all q of importance in the integration leads to

$$\lambda = \left(\frac{\sqrt{3}\, v_0}{4\tilde{\omega}_g}\lambda_L^{\,2}\right)^{1/3}. \qquad (6)$$

(Specular surface reflection reduces this by a factor of 8/9.) Pippard's expression[3] in the same limit (λ_∞) is

$$\lambda = \left(\frac{\sqrt{3}}{2\pi}\xi_0 \lambda_L^{\,2}\right)^{1/3}, \qquad (7)$$

where ξ_0 is his coherence length. Equating (6)

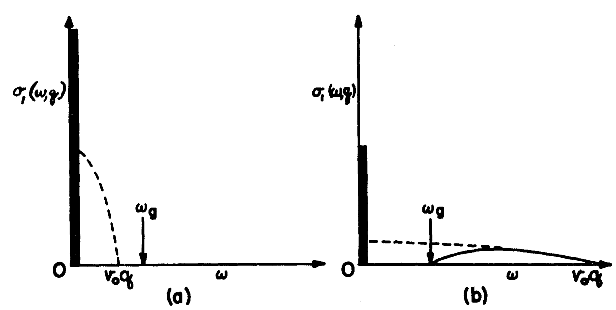

FIG. 1. Effect of the superconducting transition on the frequency-dependent conductivity for (a) long- and (b) short-wavelength transverse electromagnetic waves. The normal-state conductivity is indicated by dashed curves and extends to the maximum frequency of $v_0 q$, where v_0 is the Fermi velocity and q is the wave number. In (a) the wavelength is sufficiently long that the maximum absorption frequency in the normal state falls short of the energy gap threshold $\hbar\omega_g$. Consequently essentially all of the oscillator strength is absorbed by the delta function at zero frequency, leading to a full London current. In (b) the shorter wavelength causes the absorptions in the normal state to be spread over a frequency interval much larger than the energy gap. The strength of the delta function is therefore less and the London current is weakened. This dependence of the London current on wavelength is equivalent to the nonlocal current-field relation of Pippard.

and (7), we find

$$\xi_0 = \frac{\pi}{2}\frac{v_0}{\tilde{\omega}_g} = \left(\frac{\pi k T_c}{2\hbar\tilde{\omega}_g}\right)\left(\frac{\hbar v_0}{kT_c}\right) = a\frac{\hbar v_0}{kT_c}, \tag{8}$$

where Pippard's constant a is given in our picture by $(\pi/2)(kT_c/\hbar\tilde{\omega}_g) \approx 0.2$ using the empirical gap data mentioned above. For comparison, Faber and Pippard[12] experimentally found $a \approx 0.15$, and BCS gives 0.18. Since λ depends on ξ_0 only through a cube root, this agreement is extremely good.

Summing up, we have obtained Pippard's result for the increase in penetration depth above the London value:

$$(\lambda/\lambda_L) = 0.65(\xi_0/\lambda_L)^{1/3}, \tag{7'}$$

with ξ_0 defined by (8). This result holds only if $\xi_0 \gg \lambda_L$, which is fairly well satisfied in most cases. It is easily shown that the fractional correction in λ resulting from the deviation of $F(q)$ from (4) for small q is positive and of order $(\lambda_L/\xi_0)^{2/3}$ [or $(\lambda_L/\xi_0)^{4/3}$ for specular reflection]. The essential point is that the low-frequency electromagnetic properties of a superconductor are immediate consequences of the application of general principles to empirical infrared absorption data. Thus, the task of a microscopic theory may be properly regarded as being the deduction of the experimentally observed energy gap in the absorption spectrum. If successful in this, it automatically yields the low-frequency supercurrent behavior, including the Meissner effect, although the deduction of the Meissner effect itself follows somewhat different lines and will be contained in a subsequent publication.

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[3] A. B. Pippard, Proc. Roy. Soc. (London) A216, 547 (1953).

[4] Bardeen, Cooper, and Schrieffer, Phys. Rev. 108, 1175 (1957).

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[6] J. Lindhard, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 28, No. 8 (1954).

[7] See Sec. 3 of A. B. Pippard, in Advances in Electronics and Electron Physics, edited by L. Marton (Academic Press, Inc., New York, 1954), Vol. VI.

[8] Although the sum rule was considered in reference 1 to be a consequence of imposing the equality $\lim_{\omega\to\infty}\omega\sigma_2 = ne^2/m$, it can also be derived explicitly from quantum mechanics by a well-known technique which involves using closure and evaluating a double commutator in two different ways. See, for example, R. Kubo, J. Phys. Soc. Japan 12, 570 (1957); and L. Landau and E. Lifshitz, Quantum Mechanics (Addison-Wesley, Reading, 1958), p. 455.

[9] R. E. Glover, III, and M. Tinkham, Phys. Rev. 108, 243 (1957); 110, 778 (1958). P. L. Richards and M. Tinkham, Phys. Rev. Lett. 1, 318 (1958); M. A. Biondi and M. P. Garfunkel, Phys. Rev. Lett. 2, 143 (1959).

[10] For this case we make the assumption that the ratio of $\sigma_1(\omega, q)$ in the superconducting state to its value in the normal state is a universal function of ω/ω_g, regardless of whether the sample under study is a thin film, or an impure or pure bulk specimen.

[11] This conclusion does not follow strictly from general principles, since it is conceivable that some of the oscillator strength could be left above the energy gap, even upon passing to the limit of long wavelengths. In this case the delta function and, in turn, the London current would have only a fraction of the full strength permitted by the sum rule. An ideal pure metal in its normal state does, however, exhibit the full London current for oscillating uniform fields, and it does not seem reasonable to suppose that the superconducting transition would weaken this current. A recent calculation of Wentzel, using Bogoliubov's formulation of the BCS theory [G. Wentzel, Phys. Rev. 111, 1488 (1958)], predicts precisely this behavior, which raises some doubts concerning the validity of the theoretical basis of the calculation. The model of M. R. Schafroth and J. M. Blatt [Phys. Rev. 100, 1221 (1955)] also suffers from this defect.

[12] T. E. Faber and A. B. Pippard, Proc. Roy. Soc. (London) A231, 336 (1955).