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PROBLEM OF GAUGE INVARIANCE IN THE THEORY OF THE MEISSNER EFFECT

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This is to reply to "a note on the theory of the Meissner effect" by Pines and Schrieffer, recently published in these Letters,[1] which criticizes my paper on the Meissner effect[2] and gives preference to a different mathematical approach. Since the method recommended by them is essentially that used by Rickayzen,[3] I want to confront Rickayzen's method with mine. Both methods are based on the same, strictly gauge invariant, Hamiltonian. The phonons are treated as real particles, coupled to the electrons by a Bloch-Fröhlich interaction with a coupling strength g.

The mathematical difficulty of the present theory of superconductivity lies in the fact that certain effects of the phonon-electron coupling must be taken into account "rigorously," i.e., without expansion into powers of g, because they involve functions nonanalytic in g as $g \to 0$ (e.g., the "energy gap"). However, it is only a "reduced" problem that is solved rigorously, and "higher order" effects are still discussed in terms of an expansion in g. This is true both for Rickayzen's paper (see reference 3, Sec. 3) and mine (reference 2, Sec. 1). One hopes that, owing to the correct choice of the "reduced problem," the remaining expansion will converge rapidly if g is small. The results of the two papers are at variance because different reductions are used, and the question is which, if any, is right.

In the absence of a rigorous derivation, one necessary criterion is supplied by the gauge invariance of the result. In order to make use of this criterion it is mandatory to leave the gauge of the magnetic vector potential unrestricted, in other words to introduce *ad hoc* a longitudinal vector potential, $\vec{A}_{\text{long}}$, and show that its effects vanish identically, for the chosen reduced problem with subsequent expansion in g. Setting $\vec{A}_{\text{long}} = 0$ (or div $\vec{A} = 0$) as Rickayzen does, means foregoing the gauge invariance test (not of the Hamiltonian but of the calculational method).

Whereas my method is designed so as to meet this test, Rickayzen's method when applied to $\vec{A}_{\text{long}}$ gives a nonvanishing current $\vec{j}_{\text{long}}$ (the coefficient has the London or BCS value as $g \to 0$). One must then conclude that Rickayzen's g expansion does not converge rapidly, at least in the case of $\vec{j}_{\text{long}}$. The question remains whether there is any reason to believe that the corresponding expansion for $\vec{j}_{\text{trans}}$ is more trustworthy.

In this context, Pines and Schrieffer mention the plasma vibrations which are, indeed, excited by $\vec{A}_{\text{long}}$ but not by $\vec{A}_{\text{trans}}$.[4] However, for applying our criterion it is not necessary to introduce any Coulomb interactions at all; the latter problem is quite difficult in itself and requires other very doubtful approximations.[5] In order to make the mathematical test as clear-cut as possible it is preferable to omit the Coulomb interactions in the Hamiltonian altogether.

Even then, there exist "collective excitations,"[6] both of longitudinal and transverse type, and one may ask whether their contribution to $\vec{j}_{\text{long}}$ might tend to cancel the wrong "leading term" in Rickayzen's theory. Examples for longitudinal collective modes are phonons clothed with quasi-particle pairs, and quasi-particle pairs bound by the phonon field.[6] Applying customary approximations (e.g., Tamm-Dancoff) to these excitations, I tried to find terms which might help to restore the correct result $\vec{j}_{\text{long}} = 0$, but I did

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not succeed. I came to the conclusion that Rickayzen's method, in the Meissner effect problem, does not provide a proper starting point for subsequent low-order approximations, and that, at any rate, his power expansions in g cannot be trusted. If high-order terms contribute essentially to $\vec{j}_{\text{long}}$, they can certainly also affect $\vec{j}_{\text{trans}}$. The "leading term" may well be equally misleading in both cases.

In view of these doubts, it seemed a decisive advantage first to transform the Hamiltonian into a manifestly gauge-invariant form [reference 2, Eq. (2) with (1) and (17)]. In this new representation, the current operator $\vec{j}$ is again obtained as a power series in g [Eq. (10)], but now $\vec{j}_{\text{long}}$ vanishes automatically, to all orders in g. This does not prove, of course, that the power series for $\vec{j}_{\text{trans}}$ converges rapidly. But Pines' and Schrieffer's criticism of this expansion is groundless and futile because it is based on a comparison with a low-order approximation that is in essence the same as Rickayzen's.[7]

[1]D. Pines and J. R. Schrieffer, Phys. Rev. Lett. 1, 407 (1958). Dr. Schrieffer kindly gave me a copy, and we discussed the subject.

[2]G. Wentzel, Phys. Rev. 111, 1488 (1958).

[3]G. Rickayzen, Phys. Rev. 111, 817 (1958).

[4]J. Bardeen, Nuovo cimento 5, 1765 (1957); P. W. Anderson, Phys. Rev. 110, 827 (1958); D. Pines and J. R. Schrieffer, Nuovo cimento (to be published).

[5]E. N. Adams, Phys. Rev. 98, 1130 (1955).

[6]Corresponding to isolated roots of a secular equation. Compare Bogoliubov, Tolmachov, and Shirkov, "A new method in the theory of superconductivity," Dubna, June, 1958, sections 3.2-4; P. W. Anderson, Phys. Rev. (to be published).

[7]To realize this, one need only follow the argumentation in reference 1 (on p. 408). Note the replacement of Ψ_α and Ψ_β by eigenfunctions of a reduced Hamiltonian, and compare with reference 3.

UNSTABLE PLASMA OSCILLATIONS IN A MAGNETIC FIELD

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This work treats the small amplitude oscillations of a fully ionized quasi-neutral plasma in a uniform time-independent externally produced magnetic field. Motions of the ions and perturbations of the magnetic field are neglected. The distribution function $f(\vec{r}, \vec{v}, t)$ for the electrons satisfies

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{r}} - \frac{e}{m}\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)\cdot\frac{\partial f}{\partial \vec{v}} = 0, \tag{1}$$

and the electric field which appears in Eq. (1) satisfies

$$\nabla\cdot\vec{E} = -\nabla^2\phi = -4\pi e\int f d^3v. \tag{2}$$

The distribution function is assumed to depart only slightly from the zeroth order distribution, and the spatial dependence of the perturbation of the distribution is assumed to be given by the factor $\exp(i\vec{k}\cdot\vec{r})$. Equations (1) and (2) are linearized and then solved by taking the Laplace transform and following the procedure of Bernstein.[1] It is found that the Laplace transform of the potential is given by

$$\phi(s) = \left[-\frac{4\pi e}{k^2\omega_c}\int g(\vec{v}, \vec{k}, s)d^3v\right]/[1 - Y(s)]. \tag{3}$$

In Eq. (3), s is the Laplace transform parameter, $\omega_c = eB/mc$ is the cyclotron frequency and $g(\vec{v}, \vec{k}, s)$ is a function related to the initial value of the perturbation of the distribution function. $Y(s)$ is given by

$$Y(s) = 2\pi i\frac{\omega_p^2}{k^2}\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}dv_z\int_0^\infty v_\perp dv_\perp \times\left\{\frac{\omega_c}{v_\perp}\frac{\partial F}{\partial v_\perp}\frac{nJ_n^2(k_\perp v_\perp/\omega_c)}{(s+ik_zv_z+in\omega_c)} + k_z\frac{\partial F}{\partial v_z}\frac{J_n^2(k_\perp v_\perp/\omega_c)}{(s+ik_zv_z+in\omega_c)}\right\}. \tag{4}$$

In Eq. (4), $\omega_p = (4\pi Ne^2/m)^{1/2}$ is the plasma frequency, k_z is the component of $\vec{k}$ along $\vec{B}$, and $k_\perp$ is the perpendicular component. Similarly, v_z is the component of $\vec{v}$ along $\vec{B}$ and $v_\perp$ is the perpendicular component. J_n is the Bessel function of order n. $F(v_z, v_\perp)$ is the zeroth order distribution. It is normalized so that its integral over all of velocity space is unity.

We are particularly interested in zeroth order distributions which cause the denominator of Eq. (3) to vanish for some values of s which have positive real parts. That is,

$$Y(s) = 1 \quad \text{for } \mathrm{Re}(s)>0. \tag{5}$$

If Eq. (5) is satisfied then there will exist plas-