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NONPERTURBATIVE CALCULATION OF NUCLEAR MATRIX ELEMENTS FROM TWO-NUCLEON PHASE SHIFTS*

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A method is presented for calculating two-body shell-model interaction matrix elements from scattering phase shifts, which applies to shifts of any magnitude and to tensor coupling. It is assumed that the average (shell-model) potential, expressed in two-nucleon relative coordinates, is slowly varying within the range of the free two-nucleon interaction. Examples are given for the p shell and compared with the results of other methods, some of which include Pauli corrections.

The idea that interactions between nucleons in a nuclear system might be expressed directly in terms of free two-nucleon scattering amplitudes (or phase shifts), thus avoiding the need for explicit information on the form of the interaction, was used in the original work of Brueckner, Levinson, and Mahmoud[1] on the nuclear-matter problem, and was discussed by others.[2] The approach was more or less abandoned when it was found that the effects of the nuclear medium, on account of the Pauli principle and the binding of the nucleons, might introduce large corrections which could be calculated only by using explicitly the two-nucleon interaction.[3]

For calculating spectra of finite nuclei, it is possible that some of these difficulties are less serious. For "valence" nucleons, the Pauli principle has less effect, and some of the binding corrections are in the effective single-particle potential. Thus it might be possible to obtain interaction energies for nuclear spectra, to a good first approximation, from free-scattering phase shifts, although this approach might not be equally good for total binding energies.

In the harmonic-oscillator shell model, the interaction energy is calculated in terms of a set of matrix elements $\langle n'l'sj|t|nlsj\rangle$ of a reaction matrix or effective interaction between harmonic-oscillator states for relative motion of two nucleons. Kallio[4] has discussed an approximation which gives diagonal $(n', l'=n, l)$ reaction matrix elements in terms of the free two-nucleon phase shifts $\delta_l(E)$, evaluated for certain fixed values of the (free) relative energy, E. More recently, Elliott, Mavromatis, and Sanderson[5] have reported a somewhat different method for calculating the reaction matrix elements, which require energy averages of $\tan\delta_l(E)$. Both approaches are based on perturbation notions and are restricted to small phase shifts: Kallio's by higher order corrections in his long-range interaction v_l, and that of Elliott, Mavromatis, and Sanderson by explicit dependence on the assumption that the entire interaction is weak enough for perturbation methods.

In this paper we present another method for calculating the interaction energy for relative motion of a nucleon pair in a nucleus directly from two-nucleon phase shifts. The method is not restricted to small phase shifts and can therefore be applied to all partial waves, including tensor-coupled waves. The two-nucleon interaction is not treated as a perturbation; it is assumed to be strong and short ranged, as are the current phenomenological potentials, e.g., the Hamada-Johnston potential,[6] but its

explicit form is not needed. The restriction on the average potential which defines the unperturbed (shell-model) states of relative motion is that it be essentially constant within the range of the two-nucleon interaction. This is the case for the harmonic-oscillator potential; in fact, it is the flatness of the potential near the origin which allows Kallio to make his asymptotic approximation. This property also obtains for the square-well and Woods-Saxon potentials, to which the present method can also be applied.

To the extent that we can ignore the Pauli principle and self-consistent binding effects, the calculation of diagonal reaction matrix elements is equivalent to finding the eigenvalue shifts

$$\Delta E_{nlsj} \equiv E_{nlsj} - E_{nl}{}^{0} = \langle nlsj | t | nlsj \rangle \qquad (1)$$

of the radial Schrödinger equation in relative coordinates,

$$\left\{ -\frac{1}{M}\frac{d^2}{dr^2} + \frac{l(l+1)}{Mr^2} U(r) + V(r) - E_{nlsj} \right\} u_{nlsj}(r) = 0, \qquad (2)$$

where E^0 is the "unperturbed" eigenvalue, with no two-nucleon interaction, i.e., $V(r) = 0$. M is the nucleon mass.

We assume that $V(r)$ is of short range compared with the average potential $U(r)$, and that $U(r)$ is almost constant within the range of $V(r)$ (we take the constant to be zero). Consider the following three domains of the radius r: A, within the range of $V(r)$; B, outside the range of $V(r)$, but within the region of constant $U(r)$; and C, outside A and B. The boundaries are not unique. Now, to a good approximation a given solution $u_{nlsj}(r)$ of (2) will coincide with some solution $v_{lsj}(E, r)$ of the free scattering equation [Eq. (2) with $U(r) = 0$], in domains A and B. In domain B, however, this scattering solution is of the form

$$v_{lsj}(E, r) = Z_{lsj}(kr) \propto r j_l(kr) - r n_l(kr) \tan\delta_l \qquad (3)$$

with energy $E = k^2/M$ still to be determined. In domains B and C, $u_{nlsj}(r) = w_{nl}(E, r)$, where w_{nl} is a solution to Eq. (2) with $V(r) = 0$, at the same energy E, which goes to zero at large distances. The eigenvalue is found by matching $Z_{lsj}(r)$ and $w_{nl}(E, r)$ at some point, r_0, within the middle domain, B. Note that these two functions do not depend explicitly on $V(r)$, since they are both determined outside its range. If the two functions are matched at $r = r_0$, they will still be quite similar for any other point in A or B, since this is the region of $U(r) \simeq 0$. Therefore the eigenvalue is quite insensitive to the value of r_0, which may even be taken well within the range of $V(r)$, even though the matched functions are not the solutions of (2) at such a point.

Let us now consider the harmonic-oscillator potential, $U(r) = \frac{1}{2}\hbar\omega x^2$, $x = r/b$, with $b = (2\hbar/M\omega)^{1/2}$. We need the solutions for arbitrary E which we write following Niblack and Nigam[7] (not normed):

$$w_{nl}(E, r) = \{x^{l+1} s_l(a, x) + B_l x^{-l} t_l(a, x)\} \exp(-\tfrac{1}{2}x^2)$$

with

$$s_l(a, x) = M(\tfrac{1}{2}(a+l) + \tfrac{3}{4}, \tfrac{3}{2} + l, x^2),$$

$$t_l(a, x) = M(\tfrac{1}{2}(a-l) + \tfrac{1}{4}, \tfrac{1}{2} - l, x^2)$$

and $a = -E_{nlsj}/\hbar\omega$, where $M(a, b, z)$ is the confluent hypergeometric function,[8] and

$$B_l = -\frac{\Gamma(\frac{1}{2}(a-l) + \frac{1}{4})\Gamma(\frac{3}{2} + l)}{\Gamma(\frac{1}{2}(a+l) + \frac{3}{4})\Gamma(\frac{1}{2} - l)}. \qquad (4)$$

Note that these are not the Laguerre solutions, regular at the origin, unless $a = -(2n + l + \frac{3}{2})$. If we choose the matching radius to be small, $r_0 \ll b$, then $s_l \simeq t_l \simeq 1$. Using the small-radius forms of j_l and n_l in (3), we obtain the matching condition

$$B_l = (2l+1)!!(2l-1)!!(kb)^{-(2l+1)} \tan\delta_l(k) \qquad (5)$$

which must be solved for k (or $E_{nlsj} = k^2/M$).

The case of tensor-coupled partial waves can be treated in much the same way. One obtains two simultaneous matching equations similar to (5), for $l = J \pm 1$, in which $\tan\delta_l$ is replaced by an expression containing the eigenphase shifts δ_1, δ_2 and mixing parameter ϵ, as well as an unknown parameter, α, which gives the relative amplitudes of the two eigensolutions of the scattering problem in the desired solution of the tensor generalization of Eq. (2). This parameter α is not the same as ϵ, which gives the relative amplitudes of the two partial waves in each of the scattering eigensolutions. The coupled equations are solved for k and α.

For the case of small phase shifts, Eqs. (3)-(5) lead to a perturbation formula

$$\Delta E_{nlsj} \simeq -\hbar\omega d_{nl} \tan\delta_l$$

with

$$d_{nl}=\frac{2\Gamma(n+l+\frac{3}{2})}{\pi n!}\left(\frac{2}{kb}\right)^{2l+1},$$
$$kb=(4n+2l+3)^{1/2}, \quad (6)$$

which gives Kallio's form[4] for $l=0$.

As an example, we have calculated ΔE_{nlsj} appropriate to nuclei in the first p shell. These are listed in the first column of Table I, for $\hbar\omega=15$ MeV. The P and D states are calculated from (6) with $\tan\delta_l \simeq \delta_l$, and the S states from (5). There is a negligible effect from the mixing of δ_1 and δ_2 in $^3S_1$. This does not mean that the tensor effects are weak; they already appear in the eigenphase shift. The results are given for $r_0=0$; the results for $r_0=1.64$ F differ by $\lesssim 10\%$.

We have used the phase shifts given by Hamada and Johnston[6] so that we can compare our results with those of other reaction matrix calculations using this or similar potentials. In Table I we have listed ΔE_{nlsj} calculated by McManus and Grillot[9] and by Kim,[10] using different numerical methods to solve (2) directly. It is to these results that the present method should be a good approximation. The other columns include results by Hull and Shakin,[11] Becker and MacKellar,[12] and Kuo and Brown,[13] who have used reaction-matrix methods which take approximate account of the Pauli principle and of binding.

We note the rather close agreement of our results with those of McManus and Grillot. Kim's results differ from these for $^3S_1$ states, which may result from poor convergence of

Table I. ΔE_{nlsj} (in MeV) for $0p$ shell.

State	Present Theory $\hbar\omega$ = 15	McManus-Grillot $\hbar\omega$ = 14.5	Kim $\hbar\omega$ = 15.27	Hull-Shakin $\hbar\omega$ = 15	Becker-MacKellar $\hbar\omega$ = 15.5	Kuo-Brown $\hbar\omega$ = 14
$^1S_0$ (0s)	-8.55	-8.1	-8.31	-9.17	-7.53	-5.61
$^1S_0$ (1s)	-4.8	-4.9	-4.03	-8.00	-5.16	-4.53
$^3P_0$	-1.59	-2.0		-1.75	-1.97	-1.65
$^3P_1$	+1.73	1.8		3.32	2.51	1.99
$^3P_2$	-1.46	-2.2		-1.07	-1.28	-0.65
$^1D_2$	-0.56	-0.6		-0.64	-0.70	
$^3S_1$ (0s)	-16.1	-15.8	-8.96		-12.60	-9.73
$^3S_1$ (1s)	-8.25	-8.7	-3.80		-9.98	-8.44
$^1P_1$	1.53	1.6		6.23	2.20	1.91
$^3D_1$	1.86	2.1			1.67	
$^3D_2$	-2.35	-2.7		-2.49	-2.89	
$^3D_3$	-0.56			0.10	-0.074	

Table II. ΔE (in MeV) for P states ($\hbar\omega = 8.5$ MeV).

	$^3P_0$		$^3P_1$		$^3P_2$	
	Present theory	Elliott	Present theory	Elliott	Present theory	Elliott
0	−0.92	−1.36	0.67	0.88	−0.44	−0.49
1	−0.93	−1.25	1.03	1.16	−0.88	−0.89
2	−0.70	−0.82	1.31	1.45	−1.20	−1.17

his matrix method, for tensor coupling. The present method agrees with reaction-matrix methods about as well as they agree with each other. To correct for the omission of Pauli and binding effects, knowledge of the wave function, and therefore, of the two-nucleon interaction, would be necessary. It seems likely that the present method is accurate in the treatment of the tensor force, which has caused some trouble in reaction-matrix calculations.

Lastly, we compare our calculation of ΔE_{nlsj} for the states $^3P_{0,1,2}$ with those of Elliott, Mavromatis, and Sanderson (Table II of Ref. 5), both for $\hbar\omega = 8.5$ MeV, in Table II. The results are qualitatively similar, which suggests that there may be a weak pseudopotential, which satisfies the requirements of the theory of Elliott, Mavromatis, and Sanderson, and which approximately gives the low-energy P phase shifts in the Born approximation.

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VARIABILITY OF ELEMENTARY CHARGE AND QUASISTELLAR OBJECTS

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In my recent paper,[1] I made a suggestion that Dirac's numerical relation

$$\frac{e^2}{\gamma M^2} = 1.24\times 10^{36} = t_{\text{pres.}}, \qquad (1)$$

where e is the elementary charge, γ the gravitational constant, M the mass of a nucleon, and where $t_{\text{pres.}}$ is the present age of the universe expressed in elementary time units λ/c, may be interpreted by the assumption that, while γ remains constant, e^2 increases proportionally to t. I have suggested that this possibility may be tested by observing the value of the fine structure constant $\alpha = 2\pi e^2/hc$ in the distant galaxies. When making this suggestion, I was unaware that the test had already been made by Bahcall, Sargent, and Schmidt in their studies of the absorption spectrum of $3C$-191.[2] At the end of that paper they write the following:

"We find that: $\alpha(z = 1.95)/\alpha(z = 0) = 0.94$, 0.97, and 1.01, respectively, for the Si II lines near λ1260 and λ1527 and the Si IV lines near λ1394. We conclude that $\alpha(z = 1.95)/\alpha(z = 0) = 0.98 \pm 0.05$."

This indicates that although all lines of the spectrum are lengthened by a factor 2.945 ± 0.001, the separation between the fine-structure components of three doublets remains constant within 5%. The interpretation of this result is, however, somewhat uncertain due to the fact that there is still no general agreement concerning the nature of the celestial object in