V_2 is applied as a perturbation to the system, and it is a rather unexpected result on the basis of the JS theory. This result, however, is consistent with more recent work by Stevens and Van Eekelen on the basis of a slightly different model, and the discrepancy can possibly be attributed to some of the approximations in the JS theory.[3]

We note here that we have assumed that we can treat the terms in V_2 by perturbation theory and that this fact sets some limits to the magnitude of the coupling constant ϵ_k.

It follows that, as far as the present model is concerned, we should not try to extrapolate the results we have found to a region of too strong spin-lattice coupling, where our perturbational treatment of V_2 would in principle break down.

We would also like to draw attention to the particular form of the operators which generate the eigenfunctions from the ground state. Their phononlike characters are mixed to spinlike characters for different values of k, in such a way that one of them starts at $k \sim 0$ being a phonon creation operator; then, proceeding to higher values of k, they gradually exchange their roles to end up at the Brillouin boundaries, the first being a purely spin operator and the second, a purely phonon one.

The next step would be to solve the problem for subspaces with more than one excitation. One would try to get other eigenfunctions by repeated applications of the operators $f_k \alpha_k^\dagger + g_k S_{k+}$ to the ground state.

Unfortunately this is not the case: States of the form

$$|a\rangle = \Pi_k \{(a_k!)^{-1/2} (f_k \alpha_k^\dagger + g_k S_{k+}) |0\rangle\},$$

where the a_k are natural numbers, are not in general eigenstates of our Hamiltonian. Here we seem to run into very much the same difficulties as those encountered in the theory of ferromagnetic spin waves.[4]

We are indebted to Professor K. W. H. Stevens and Dr. H. A. M. Van Eekelen for the kind communication of unpublished results and for much appreciated comments on this manuscript.

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MAGNETOELASTIC EFFECTS AND THE MAGNETIC PROPERTIES OF RARE-EARTH METALS

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(Received 30 August 1967)

We show that the driving force for the spiral-to-ferromagnet transition in Dy and Tb is the energy of cylindrical symmetry associated with the lowest order magnetostriction effects. We also point out that the experimental temperature dependence of the spin-wave energy gap for Tb does not agree with that calculated using the frozen-lattice approximation suggested by Turov and Shavrov for magnetoelastic effects.

Both dysprosium and terbium initially order with spiral arrangements, and with decreasing temperature have abrupt transitions (at $T_C = 221$°K for Tb, 85°K for Dy) to ferromagnetic ordering with appreciable lattice distortion ($\delta l/l \sim 10^{-2}$–10^{-3}). The question of the driving mechanism for this transition is particularly relevant because of recent neutron-scattering studies[1] of spin-wave spectra of Tb and Tb-Ho alloys. The possibility of magnetoelastic effects playing an important role in the spiral-to-ferromagnet transition was pointed out by Enz[2] at an early stage, but such effects have often not been considered in subsequent studies. The present note presents quantitative arguments that the driving force for the spiral-to-ferromagnetic transition in Dy and Tb is the energy of cylindrical symmetry associated with the lowest order magnetostriction effects. This is in contrast with other possibilities that may be considered, e.g., explicit temperature dependence of the exchange

mechanism, or the planar crystal-field anisotropy of the unstrained crystal. Basically, the spiral arrangement serves to restrain ("clamp") each successive plane along the c axis from developing the strain that would minimize the combined elastic and magnetoelastic energy. Transition to a ferromagnet allows such energetically favorable strains to develop.

Our study then indicates that the combination of the presence of a clearly defined maximum in $J(\vec{q})$ (the Fourier-transformed exchange energy) in the spiral regime in a Tb-Ho alloy for a $\vec{q}$ corresponding to the spiral period and the considerable decrease or possible absence of such a maximum in the ferromagnetic[1] regime, is the result of the transition (i.e., the change in lattice structure), and is not the driving mechanism for the transition. Conceptually, $J(\vec{q})$ could retain its maximum at some incommensurate $\vec{q}$ in the ferromagnetic regime.

We have also examined the expected effect of magnetoelastic forces on the spin waves. The experimental[1] temperature dependence of the spin-wave energy gap for Tb does not agree with that calculated using the frozen-lattice approximation suggested by Turov and Shavrov[3] for magnetoelastic effects. Rather, the behavior found suggests that the strains move with the motion of the magnetization for the long-wavelength modes.

For Dy, an analysis by Elliot[4] gives the experimental values shown in Fig. 1 for the term in the free energy which tries to drive the system ferromagnetic in the spiral temperature range between 85 and 179°K. We now show that this energy is closely equal to the energy decrease associated with the magnetostriction for ferromagnetic Dy.

The equilibrium values of the strains arising from a given net magnetization are determined by minimizing the total strain-dependent energy density E_{ms} (elastic plus magnetoelastic energy) with respect to each of the strain components. Considering strains developed for magnetization in the plane, the equilibrium energy associated with magnetostriction for ferromagnetic Dy is then given by

$$E_{ms} = -\tfrac{1}{8}c^{\gamma}(\lambda^{\gamma})^2. \qquad (1)$$

Here λ^{γ} is the magnetostriction coefficient representing the distortion of the circular symmetry of the basal plane by the rotation of the component of magnetization in the plane, and

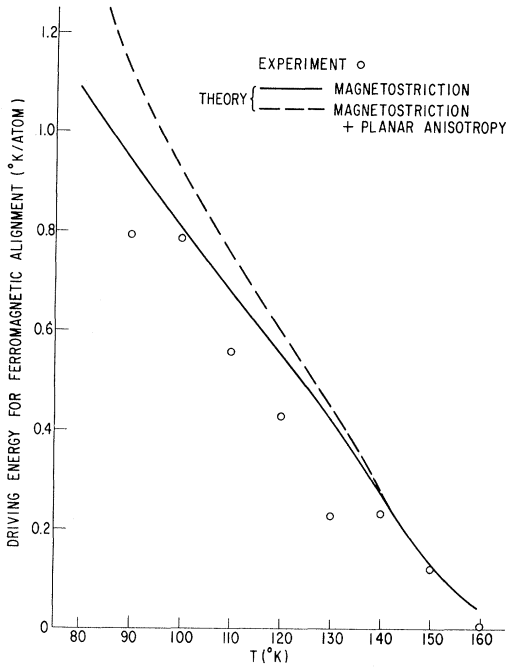

FIG. 1. Temperature dependence of driving energy for ferromagnetic alignment in Dy.

c^{γ} is the appropriate symmetry combination of elastic constants.[5]

The theory of Callen and Callen[5] gives the temperature dependence expected for λ^{γ}:

$$\lambda^{\gamma}(T) = \lambda^{\gamma}(0)\hat{I}_{5/2}[\mathcal{L}^{-1}(\sigma)]. \qquad (2)$$

Here $\hat{I}_{5/2}(x)$ is the normalized hyperbolic Bessel function, and $\mathcal{L}^{-1}(\sigma)$ is the inverse Langevin function of the reduced magnetization σ. For low temperatures, $\hat{I}_{5/2}[\mathcal{L}^{-1}(\sigma)] \sim \sigma^3$. This proportionality is reasonably good throughout the ordered region, although as $\sigma \to 0$, $\hat{I}_{5/2} \sim \sigma^2$. Clark, DeSavage, and Bozorth[6] and Rhyne and Legvold[7] find excellent agreement with Eq. (2) for Dy and Tb, respectively. Thus, the energy difference between spiral and ferromagnetic arrangements given by (1) goes approximately as σ^6. The solid curve in Fig. 1 shows this contribution to the driving energy calculated from (1) using (2) and the experimental values[6-8] of parameters at $T = 0$ (regarding c^{γ} as temperature independent). The agreement with experiment is excellent.

There is a second contribution to the driving energy for the spiral to ferromagnet transition. As discussed by Elliott,[4] this is the decrease in the hexagonal planar anisotropy energy of the undistorted lattice. Elliott at-

tributed a σ^6 theoretical temperature dependence to this contribution; however, Callen and Callen[9] have shown that a planar anisotropy corresponding to a sixth-order spherical harmonic and an axial term of the same order have the same σ^{21} temperature dependence at low T. This contribution to the driving energy per atom then equals $P_6{}^6S^6\hat{I}_{13/2}[\mathcal{L}^{-1}(\sigma)]$, where $\hat{I}_{13/2}\sim\sigma^{21}$ through most of the magnetically ordered regime. At $T=0$ for Dy, $\bar{E}_{ms}=-2.0$°K/atom and $P_6{}^6S^6=-2.4$°K/atom.[10] Thus, at $T=0$, the two contributions to the driving energy are comparable. However, the planar anisotropy energy drops off much more rapidly with temperature, so that, as shown in Fig. 1, for temperatures approaching and exceeding T_C, the magnetostriction term in the driving energy dominates. Clearly, whichever of the two contributions is dominant in assisting the applied field in bringing about the transition for T just above T_C is dominant in driving the spontaneous transition at T_C. One can also obtain experimental values of $\bar{E}_{ms}(0)$ and $P_6{}^6S^6$ for Tb from the work of Rhyne and Legvold[8] and of Rhyne and Clark,[11] respectively. This gives, for Tb, $\bar{E}_{ms}(0)=-1.97$°K/atom and $P_6{}^6S^6=-0.57$°K/atom. Hence the dominance of the magnetostriction contribution to the driving energy should occur at even higher relative magnetization for Tb than in Dy.

Turov and Shavrov[3] have suggested some very provocative ideas for the magnetoelastic effects on the spin-wave energy gap at $q=0$ for the ferromagnetically aligned heavy rare-earth metals. Turov and Shavrov suggest that the correct way to find the uniform mode frequency is to regard the strain as frozen at its equilibrium position. Then in the excited state, the relative orientation of moment and strain changes, and there is a net increase of energy relative to the equilibrium state even though the equilibrium energy associated with the magnetostriction has cylindrical symmetry. Using the macroscopic equation of motion technique, they estimated this effect as giving an energy gap at low T of about 10°K for Dy.

We have considered these ideas in more detail by treating the equilibrium strains $\bar{\epsilon}_1{}^\gamma$ and $\bar{\epsilon}_2{}^\gamma$ as classical quantities in the spin Hamiltonian. We have then found the spin-wave spectrum by standard technique. The short-dashed curve in Fig. 2 shows the predicted temperature dependence of the $\vec{q}=0$ spin wave for Tb in the frozen-lattice approximation.

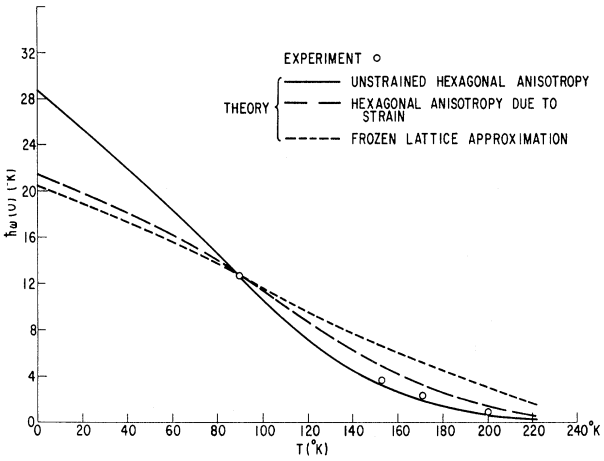

FIG. 2. Temperature dependence of $\vec{q}=0$ spin-wave energy for Tb.

This has been calculated using the experimental[11] value for $P_6{}^6$ and choosing $\lambda^\gamma(0)$ to match the neutron value[1] of $\hbar\omega(0)$ at 90°K. The solid curve on the other hand corresponds to the "ordinary" case where the planar anisotropy gives an effective field in the plane.[1] The value of $P_6{}^6$ has been chosen to match the experimental $\hbar\omega(0)$ at 90°K.

One can also consider the case where the magnetoelastic interaction contributes to the spin-wave energies, but the frozen-lattice approximation does not apply. Since the lowest order energy associated with the magnetostriction has cylindrical symmetry, it then does not contribute to the spin-wave energy. The next highest order contribution to the magnetostriction corresponds to an energy of hexagonal symmetry. Such an energy will contribute to the spin-wave energy even when the frozen-lattice approximation does not apply. The long-dashed curve in Fig. 2 applies for this approximation, where the hexagonal energy associated with magnetostriction has been chosen to match experiment at 90°K.

For all three approximations shown in Fig. 2, the temperature dependence has been found using the classical approximation valid for long wavelengths, where the effective field corresponding to a given anisotropic energy has the temperature dependence of that energy divided by the magnetization. The key point shown in Fig. 2 is that the temperature dependence for the frozen-lattice approximation is decidedly weaker than for the other two approximations. (The cylindrical magnetostriction

energy gives an effective field $\sim\sigma^5$ as opposed to σ^9 for the hexagonal magnetostriction energy, and σ^{20} for the unstrained hexagonal anisotropy energy.) The experimental[2] spin-wave energies fall between the two curves for which the strain is not frozen. (The good agreement with the unstrained-hexagonal-anisotropy calculation has already been pointed out by Møller et al.[1])

Presumably the lack of applicability of the frozen lattice approximation is related to the low value of $\hbar\omega(0)$ relative to the Debye temperature. Vibrational modes are available whose frequencies are high compared with $\omega(0)$, so that the strain can accomodate itself to the instantaneous motion of the magnetization.

The inapplicability of the frozen-lattice approximation could be demonstrated more clearly by pursuing the neutron experiments to lower temperatures. Actually, there is an even more striking way to see whether or not the frozen-lattice approximation applies. When the frozen-lattice approximation is inapplicable, a magnetic field applied along a hard hexagonal axis can reduce $\omega(0)$ to zero. On the other hand, because of the cylindrical symmetry of the lowest order magnetostriction energy, this cannot be done for the frozen-lattice case.[3] For example, at 90°K our calculations for the frozen-lattice case predict that $\hbar\omega(0)$ cannot be reduced below 10.8°K. (This occurs for 8 kOe along a hard axis.) The use of such a field in the neutron experiments could unambiguously define the applicability of the frozen-lattice approximation. Actually, a field has been used in this manner in resonance experiments. The results of these experiments[12-13] also indicate the inapplicability of the frozen-lattice approximation to Dy and Tb. This, together with details of the present treatment, will be discussed in a future publication.

Thus, the main point of this note is that magnetoelastic interactions have a strong qualitative effect on the equilibrium properties of heavy rare-earth metals. On the other hand, the excited-state behavior is probably qualitatively similar to that in the absence of such interactions. The difference in the two types of behavior rests on the cylindrical symmetry of the lowest order energy contributing to the magnetostriction.

I am grateful to Dr. H. Bjerrum Møller, Dr. J. C. Gylden Houmann, and Professor A. R. Mackintosh for making available the experimental results for the temperature dependence of the spin-wave energies in Tb. I wish to thank E. Kreiger and A. Warner for their aid in the numerical calculations.

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