ANALYSIS OF πN SCATTERING USING PARTIAL-WAVE DISPERSION RELATIONS IN THE W PLANE*

Philip W. Coulter and Gordon L. Shaw
University of California, Irvine, California
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Hamilton and co-workers[1] have made extensive use of partial-wave dispersion relations in the s (square of the total energy W in the c.m. system) plane in analyzing πN scattering to obtain a number of interesting results. We have performed similar calculations for the $I=\frac{1}{2}, J=\frac{1}{2}$ partial waves in the W plane and found some striking features which reflect on a variety of questions.

The partial-wave dispersion relation used by Hamilton is

$$f(s)=P(s)+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}ds'\frac{\operatorname{Im}f(s')}{s'-s}+\frac{1}{\pi}\int_{0}^{(m-\mu)^2}ds'\frac{\operatorname{Im}f(s')}{s'-s}+\Delta(s). \quad (1)$$

The term $P(s)$ denotes the contribution of nucleon exchange to the short cut, $(m-\mu^2/m)^2 \leqslant s \leqslant m^2+2\mu^2$, and the direct nucleon pole at $s=m^2$ for the P_{11} amplitude.[2] The first integral is over the unitarity cut whereas the second integral is over an unphysical region and must be computed from crossing symmetry. The lower limit on the latter integral is 0 and not $-\infty$ because the square-root cut in s which one encounters in πN scattering when working in the s plane is not easily taken into account. The term $\Delta(s)$, called the discrepancy, accounts for the cut from $-\infty$ to 0 (including effects of the square-root cut) and the circular cut of radius $m^2-\mu^2$ centered at $s=0$ which is due entirely to the t channel ($\pi\pi\rightarrow N\bar{N}$).

The procedure is to evaluate f and the first three terms on the right-hand side of (1) from phase shifts, known masses, and coupling constants and thus determine $\Delta(s)$ at low energy. The conclusion[1] from the S-wave dispersion relations was that $\Delta(s)$ could be accounted for by a faraway contribution (almost constant) plus a rapidly varying part due to the $J=1, I=1$ and the (large) $J=0, I=0$ $\pi\pi\rightarrow N\bar{N}$ terms dominated by the ρ resonance and a strongly attractive S-wave $\pi\pi$ state "σ," respectively.

We have done similar calculations in the W plane (which for a given J couples the states $l=J\pm\frac{1}{2}$) and find some striking features in the $I=\frac{1}{2}, J=\frac{1}{2}$ amplitude. The partial-wave dispersion relations in the W plane for the $I=\frac{1}{2}, J=\frac{1}{2}$ amplitude are [using the relation $f_S(W)=-f_P(-W)$][3]

$$f_S(W)=B_S(W)+\frac{1}{\pi}\int_{m+\mu}^{\infty}\left[\frac{\operatorname{Im}f_S(W')}{W'-W}-\frac{\operatorname{Im}f_P(W')}{W'+W}\right]dW', \quad (2)$$

$$f_P(W)=B_P(W)+\frac{1}{\pi}\int_{m+\mu}^{\infty}\left[\frac{\operatorname{Im}f_P(W')}{W'-W}-\frac{\operatorname{Im}f_S(W')}{W'+W}\right]dW', \quad (3)$$

where B contains the generalized potential terms arising from the unphysical cuts plus the direct nucleon pole which appears in both the S- and P-wave relations (2) and (3).

The amplitudes are defined by

$$f_i=[\exp(2i\delta_i)-1]/2i\rho_i, \quad i=S,P,$$

where, as discussed by Frautschi and Walecka,[4] the appropriate kinematical factor is

$$\rho_S(W)=(E+m)(k/W),$$

$$\rho_P(W)=(E-m)(k/W),$$

with E the energy of the nucleon.

We can now compute the potential terms in B directly from known u- and t-channel reactions without complications due to square-root cuts. We assume initially that this can be done approximately by considering ρ, N, and $N^*(1238)$ exchange in the crossed channels as had been done in previous N/D calculations.[5-7] Our discrepancy, $\Delta_l(W)$, is defined as the quantity which must be added to this approximate potential term in order to make (2) and (3) valid.

We have evaluated the integrals in (2) and (3) up to pion laboratory kinetic energies, E_L, of 1 BeV using recent complex phase-shift analyses which extend up to this energy. However, we find that although the integrals are well convergent, the contribution above 1 BeV can be larger than that up to 1 BeV. A crude estimate of the error involved in neglecting the remaining parts of the integrals is indicated in Tables I and II. The uncomputed part of the integral clearly cannot contribute much to the variation in $\Delta(W)$ over this energy range and in fact does not contribute much to the magnitude of $\Delta(W)$. The integrals are computed by using the phase shifts determined by Roper[8] up to $E_L = 350$ MeV and then using phase shifts taken from Bareyre et al.[9] up to 1 BeV. The contribution from the range 350-1000 MeV is generally larger than the 0- to 350-MeV contribution in both S- and P-wave scattering.

The results of the calculations are summarized in Tables I and II for the S_{11} and P_{11} partial waves, respectively. We would like to emphasize the following points:

(1) The conventional (N, N^*, and ρ) potential terms are obviously inadequate to explain the phase shifts.

(2) By working in the W plane we pick up a contribution to the S_{11} potential term from the direct nucleon pole (at $W=-m$). This exactly known contribution is larger in magnitude than the ρ, N, and N^* exchange terms. Thus, a simple modification (e.g., a suppression of the u-channel forces) will not explain $\Delta_{S_{11}}$. (On the other hand, the u-channel forces are comparable in magnitude with $\Delta_{P_{11}}$.) It would seem that an investigation of the exchange potentials in terms of Regge trajectories would be profitable.[10]

(3) Following Hamilton et al.,[1] we fit the energy variation of Δ_S by introducing an $I=0, J=0$ t-channel potential term B_σ dominated by an $l=0, I=0$ $\pi\pi$ resonance "σ." We set the mass of the σ equal to 350 MeV and adjust the over-all strength $g_{\sigma N\bar{N}}g_{\sigma\pi\pi}$ to get an excellent fit to the s dependence of Δ_S (as seen in the last column of Table I). This determines the term B_σ for the P wave and as seen in Table II this large, rapidly varying term would make the P-wave discrepancy much worse. On the other hand, if we use the value $g_{\sigma N\bar{N}} \approx 1.7$ determined by Scotti and Wong[11] in their analysis of NN scattering, and estimate $g_{\sigma\pi\pi}$ by assuming a 100-MeV width for the σ, we obtain terms

Table I. S_{11} fit to the dispersion relation [Eq. (2)]. I_S and I_P refer to the integrals over the S_{11} and P_{11} unitarity cuts up to 1 BeV, respectively; the numbers shown in parentheses are the contributions to the integrals between 1 and 2 BeV assuming that the unitarity limit is maintained in this interval. B_N, B_ρ, and B_{N^*} are the contributions to the potential term from the nucleon, ρ meson, and N^*(1238 MeV) resonance. B is the sum of these contributions and Δ is the discrepancy, computed from $\mathrm{Re}f = B + \Delta + I_S - I_P$ where I_S is a principal-value integral. The potential terms are those used by Ball and Wong [Ref. (6)]; in their notation the coupling constants are $g^2/4\pi = 14.6$, $\gamma_{33} = 0.06$, $\gamma_1 = -1$, and $\gamma_2 = -0.27$. B_σ is the possible contribution from an $l=0$, $T=0$ $\pi\pi$ resonance (at 350 MeV) where the over-all coupling $g_{\sigma NN}g_{\sigma\pi\pi}$ has been adjusted to fit the energy variation in $\Delta_{S_{11}}$ which then fixes the contribution of B_σ to the P_{11} potential term shown in Table II.

W	E_L (MeV)	$\mathrm{Re}f_{S_{11}}$	I_S	I_P	B_N Direct	B_N Exchange	B_ρ Electric	B_ρ Magnetic	B_{N^*}	B	$\Delta_{S_{11}}$	B_σ
7.72	3.8	0.104	0.041 (0.030)	0.105 (0.072)	−1.52	0.589	0.105	−0.0002	−0.517	−1.34	1.51	0.51
8.30	100	0.078	0.038 (0.033)	0.102 (0.070)	−1.46	0.602	0.152	−0.007	−0.530	−1.24	1.38	0.38
8.58	150	0.073	0.039 (0.035)	0.101 (0.069)	−1.43	0.607	0.172	−0.013	−0.535	−1.20	1.34	0.33
8.86	200	0.070	0.042 (0.037)	0.099 (0.068)	−1.41	0.612	0.189	−0.019	−0.540	−1.17	1.29	0.30
9.38	300	0.068	0.051 (0.041)	0.097 (0.067)	−1.36	0.619	0.217	−0.033	−0.546	−1.11	1.23	0.24

Table II. P_{11} fit to the dispersion relation [Eq. (3)]. The notation is the same as in Table I. Δ is computed from the relation $\mathrm{Re}f = B + \Delta + I_P - I_S$ where I_P is a principal-value integral.

W	E_L (MeV)	$\mathrm{Re}f_{P_{11}}$	I_S	I_P	B_N Direct	B_N Exchange	B_ρ Electric	B_ρ Magnetic	B_{N^*}	B	$\Delta_{P_{11}}$	B_σ
7.72	3.8	−9.05	0.006 (0.008)	0.780 (0.256)	−21.4	2.74	1.91	4.16	8.58	−3.98	−5.85	9.23
8.30	100	−2.99	0.006 (0.008)	0.859 (0.285)	−13.7	1.87	2.01	4.21	8.38	2.81	−5.66	5.00
8.58	150	−0.978	0.005 (0.008)	1.04 (0.300)	−11.6	1.65	2.02	4.19	8.33	4.56	−6.58	3.84
8.86	200	1.00	0.005 (0.008)	1.53 (0.318)	−10.3	1.47	2.02	4.16	8.31	5.80	−6.33	3.00
9.38	300	3.88	0.004 (0.008)	2.01 (0.355)	− 8.18	1.25	2.00	4.07	8.31	7.45	−5.58	1.99

$B_\sigma(W)$ which are an order of magnitude smaller than those given in Tables I and II. We conclude that the σ does not contribute significantly to πN scattering in $J=\frac{1}{2}$ partial waves. (However, it should be important in higher partial waves.)

(4) We have seen that our knowledge of the potential terms B (at least for the $I=\frac{1}{2}, J=\frac{1}{2}$ states) is quite poor. Furthermore, the amount of error introduced in the S-wave dispersion relation due to a lack of knowledge of the phase shifts above 1 BeV is potentially large enough to make this dispersion relation unreliable as a test (in performing phase-shift analyses) for choosing between various sets of πN phase shifts above 300 MeV.

(5) Previous (single-channel) N/D calculations[5-7] have tried to determine the nucleon as a πN bound state. Taking the potential terms B_N exchange, B_{N^*}, and B_ρ as input, a cutoff was adjusted to give a bound state at the position of the nucleon. Not only are the calculated phase shifts completely wrong, but the calculated residue gave $g^2/4\pi \gtrsim 26$ (as compared with the correct value 14.6) and the bound state was produced over a quite limited range of values of the cutoff. We have investigated whether improved potential terms would influence this situation: We used simple expressions to approximate the Δ's and added these to the potential terms B_N exchange, B_{N^*}, and B_ρ. This new B was used as input into the N/D equations and the cutoff adjusted to produce the nucleon as a bound state. A bound state was easily produced over a wide range of values of the cutoff and furthermore we obtained residues at the position of the nucleon giving values of $g^2/4\pi$ as small as 19 when inelastic effects were included by using the Frye-Warnock equations.[12] The calculated phase shifts were still wrong. Clearly, a two-channel calculation is needed to obtain both the nucleon as a bound state and the Roper resonance.[13]

(6) Current-algebra[14] calculations for the S-wave πN scattering lengths have been in excellent agreement with experiment. Sakurai[15] has shown that the current-algebra predictions can be obtained by the ρ-dominance model. We see from Table I that the potential term for ρ exchange is indeed very nearly equal to the real part of the amplitude at threshold. However, B_ρ has the wrong energy dependence to explain the S_{11} effective range. The assumption of ρ dominance is not adequate to explain the low-energy S_{11} phase shifts.

In conclusion, we note that the procedure of Hamilton et al.,[1] in using partial-wave dispersion relations together with experimentally determined phase shifts, is indeed a powerful technique in obtaining information about the nature of the potential terms. This approach is clearly applicable to a variety of problems.

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