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[10]One such model is readily constructed by assuming that the conserved unitary spin currents are proportional to the vector fields, and parametrizing the meson-baryon form factors in terms of a single pole. If one is unwilling to accept such models, one can appeal directly to the observed proportionality of form factors for $q^2 < 0$ and make a "smooth extrapolation"!

WHY IS $1-2F_\pi^2\gamma_{\rho\pi\pi}^2/m_\rho^2 \cong 0$? †

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(Received 24 July 1967)

It is well known that the $\rho\pi\pi$ coupling constant and ρ mass are related experimentally to F_π, the fundamental constant of partially conserved axial-vector currents (PCAC),[1] by the relation

$$1-2F_\pi^2\gamma_{\rho\pi\pi}^2/m_\rho^2 \cong 0. \tag{1}$$

This relation was first noted by Kawarabayashi and Suzuki[2] who obtained Eq. (1) by applying PCAC, ρ dominance, and current algebra. This relation has also been obtained by several authors,[3-5] using various other methods and approximations. We would like to point out, in this Letter, that Relation (1) does not follow from PCAC, ρ dominance, and current algebra as unambiguously as has previously been thought. In particular, we shall suggest a generalization of Relation (1) which indicates that its approximate validity is related to the relative weakness of the interaction in the $\rho\pi$ system with the quantum numbers of the A_1 meson, $I(J^{PG}) = 1(1^{+-})$.

Relation (1) was obtained in Ref. 2 by extrapolating $\gamma_{\rho\pi\pi}$ to zero π and ρ masses. The extrapolation of the ρ mass was not done explicitly, however, and we have not been able to find a satisfactory definition of this extrapolation which leads to the desired result. Consider, for example, the most direct definition of $\gamma_{\rho\pi\pi}$ for all three particles off their mass shells,

$$2i\epsilon_{abc}(\epsilon k)\gamma_{\rho\pi\pi}(k^2,q^2,p^2) = P_\pi(k^2)P_\pi(q^2)P_\rho(p^2)\epsilon^\lambda \int e^{ikx+iqy}\langle 0|T[D^a(x)D^b(y)V_\lambda{}^c(0)]|0\rangle, \tag{2}$$

$$D^a(x) = \partial^\mu A_\mu{}^a(x),\quad P_\pi(k^2) = (m_\pi{}^2-k^2)/m_\pi{}^2F_\pi,\quad p=k+q,\quad P_\rho(p^2) = (m_\rho{}^2-p^2)/F_\rho,$$

with $\delta_{ab}F_\rho\epsilon_\lambda = \langle 0|V_\lambda{}^a|\rho^b\epsilon p\rangle$. Hopefully, ρ dominance of the vector current $V_\lambda{}^c$ will make $\gamma_{\rho\pi\pi}$ a slowly varying function of p^2. Current algebra can be applied to Eq. (2) in the usual manner of taking the derivatives of the axial-vector currents outside the time-ordered product and taking the limit of Eq. (2) as $k,q\to 0$. A prediction for $\gamma_{\rho\pi\pi}(0,0,0)$ is obtained by equating terms linear in k and q,

$$\gamma_{\rho\pi\pi}(0,0,0) = [F_\pi(0)/F_\pi](m_\rho{}^2/F_\rho). \tag{3}$$

$F_\pi(0)$ is the extrapolation of F_π to zero pion mass. The algebra leading from Eq. (2) to Eq. (3) is straightforward but tedious. When applying the current commutation rules, one obtains both the axial-vector and vector current propagators, $\Delta_{\mu\lambda}{}^A$ and $\Delta_{\mu\lambda}{}^V$, but the covariant part of $\Delta_{\mu\lambda}{}^V$ is quadratic in k and q and does not contribute to Eq. (3). The noncovariant parts of the two propagators cancel by virtue of Weinberg's sum rule,[6] as they must if the right-hand side of Eq. (2) is to be covariant as $k,q\to 0$. One is then left with the covariant part of $\Delta_{\mu\lambda}{}^A$, with only the contribution from the π intermediate state (and 0^- continuum) giving a term linear in the momenta. Returning to Eq. (3), we see that, since we expect from PCAC that $F_\pi(0) \cong F_\pi$, this extrapolation of $\gamma_{\rho\pi\pi}$ yields the conventional result of ρ dominance of the electromagnet form factor of the pion, $F_\rho = m_\rho^2/\gamma_{\rho\pi\pi}$. Hence Eq. (2) defines a very good extrapolation of $\gamma_{\rho\pi\pi}$ but does not tell us anything new.

There are, of course, many ways of extrapolating a physical quantity off the mass shell and PCAC and ρ dominance still do not provide a unique prescription for the extrapolation. For example, Kawarabayashi and Suzuki[2] first take both pions off the mass shell, apply current algebra, and then extrapolate the result of the ρ mass shell. This leads to the result

$$2\gamma_{\rho\pi\pi}(k^2, q^2, p^2) \to F_\rho(p^2)/F_\pi^2, \quad k, q \to 0,$$

where $F_\rho(p^2)$ is the extrapolation of $F_\rho \equiv F_\rho(m_\rho^2)$. Unfortunately, the definition of $F_\rho(p^2)$ using ρ dominance,

$$\delta_{ab}\epsilon_\mu F_\rho(p^2) = P_\rho(p^2)\epsilon^\nu i\int e^{-ipx}\langle 0|T[V_\nu^a(x)V_\mu^b(0)]|0\rangle = \delta_{ab}P_\rho(p^2)\epsilon^\nu \Delta_{\nu\mu}^V(p), \tag{4}$$

is noncovariant unless $p^2 = m_\rho^2$, because of the Schwinger term in the vector current commutator. If we take the conventional covariant part of the propagator in Eq. (4), $\bar{\Delta}_{\mu\nu}^V = (p^2 g_{\mu\nu} - p_\mu p_\nu)\Delta_V(p^2)$, we find $F_\rho(p^2) \to (p^2/m_\rho^2)F_\rho$ as $p \to 0$. In other words, if the ρ had zero mass, its coupling to the photon should vanish. Because Eq. (4) is noncovariant, however, we can always define the covariant part in any way we like as long as the limit as $p^2 \to m_\rho^2$ is unchanged. In particular, if we subtract from $\Delta_{\mu\nu}^V$ only the term proportional to $\delta_{\mu 0}\delta_{\nu 0}$, then we can obtain the desired result, $F_\rho(0) = F_\rho(m_\rho^2)$, leading to Relation (1). In this case we might just as well interpret the observed validity of Relation (1) as telling us how to define the extrapolation of F_ρ.

Another way of carrying out the extrapolation of $\gamma_{\rho\pi\pi}$ is to take one of the pions and the ρ off the mass shell first, apply current algebra, and then extrapolate the remaining pion. This time the procedure is straightforward and we obtain, again, Eq. (3).

The other derivations of Relation (1) require other assumptions in addition to PCAC, ρ dominance, and current algebra. Sakurai[5] must assume that the low-energy, S-wave, pion-nucleon scattering amplitude with $I=1$ in the t channel is given by ρ exchange. Gilman and Schnitzer[4] obtain (1) by comparing the spin-averaged sum rule for π-ρ scattering obtained for zero π mass with the sum rule obtained for zero ρ mass. Relation (1) is obtained if the difference between these sum rules is saturated by the π intermediate state. While the ω and φ intermediate state contributions exactly cancel, they must assume that there are no other important interactions in the $\pi\rho$ system near threshold, such as a 1^+ meson, for example.

Riazuddin and Fayyazuddin[3] obtain (1) by deriving a Goldberger-Treiman-like relation for the matrix element,

$$i\epsilon_{abc}F_\lambda = \langle \pi^a q|iA_\lambda^b|\rho^c \epsilon p\rangle$$
$$= F_0(k^2, q^2)\epsilon_\lambda + (\epsilon q)[F_+(p+q)_\lambda + F_- k_\lambda], \tag{5}$$

extrapolated to zero π mass. As usual, $p = k + q$. But, since there are three invariant functions to deal with, PCAC and current algebra do not yield enough information to determine $\gamma_{\rho\pi\pi}$ the way Goldberger and Trieman determined $g_{\pi N}$ from $\langle N|A_\lambda|N\rangle$. Relation (1) was obtained in Ref. 3 by assuming F_0 to be constant, $F_+ = 0$, and F_- given by π dominance. Again, the derivation requires the explicit neglect of possible contributions to $F_0, F_\pm$ coming from an interaction in the $\pi\rho$ system with the quantum numbers $1(1^{+-})$. The rest of this Letter will breifly discuss the results obtained when contributions from this $\pi\rho$ state are included. For simplicity we will assume that this state can be represented by a single axial-vector meson, which we denote by A (let us keep an open mind about associating the A with the still controversial A_1 meson).

To apply current algebra and PCAC to Eq. (5), extrapolate F_λ in the pion mass,

$$i\epsilon_{abc}F_\lambda = iP_\pi(q^2)\int e^{iqx}\theta(x^0)$$
$$\times \langle 0|[\partial^u A_\mu^a(x), iA_\lambda^b(0)]|\rho^c \epsilon p\rangle. \tag{6}$$

Taking the limit $q \to 0$ of Eq. (6) leads to the condition

$$F_0(m_\rho^2, 0) = F_\rho / F_\pi. \tag{7}$$

Assuming that σ_{ab} in the commutator, $\delta(x^0) \times[A_0^b(x), \partial^u A_\mu^a(0)] = i\delta(x)\sigma_{ab}$, is a local scalar operator, or, at least symmetric in a, b, so that $\langle 0|\sigma_{ab}|\rho^c\rangle = 0$, then $k^\lambda F_\lambda$ is proportional to $\gamma_{\rho\pi\pi}$, extrapolated by PCAC off both pion mass shells:

$$k^\lambda F_\lambda = (\epsilon q)[(m_\rho^2 - q^2)F_+ + k^2 F_- - F_0]$$

$$= -2(\epsilon q)\gamma_{\rho\pi\pi}(k^2, q^2, m_\rho^2)/P_\pi(k^2). \tag{8}$$

Apply unsubtracted dispersion relations in k^2 for $F_\pm$, keeping q^2 fixed. The only intermediate states that can contribute to the dispersive parts of $F_\pm$ are those states coupled to the $\pi\rho$ system and with quantum numbers $1(0^{--})$ or $1(1^{+-})$. We approximate $F_\pm$, therefore, by taking only π and A intermediate states. This enables us to use Eq. (8) to determine $F_0(k^2, q^2)$ in terms of $f_{A\rho\pi}$ and $g_{A\rho\pi}$, the two $A\rho\pi$ coupling constants; F_A, the analog of F_ρ; and $\gamma_{\rho\pi\pi}$. Equation (7) then leads to the sum rule

$$f_{A\rho\pi}(0) - g_{A\rho\pi}(0)$$

$$= \frac{1-2\gamma_{\rho\pi\pi}(0,0,m_\rho^2)F_\pi^2}{F_\rho} \frac{m_A^2}{m_\rho^2} \frac{F_\rho}{F_\pi F_A}, \tag{9}$$

where $f_{A\rho\pi}(q^2)$, $g_{A\rho\pi}(q^2)$ are defined by

$$P_\pi(q^2)\langle \rho^a \epsilon' k | \partial^u A_\mu^b | A^c \epsilon p\rangle = i\epsilon_{abc}[(m_A^2 - m_\rho^2) f_{A\rho\pi}(-\epsilon\epsilon') - 2g_{A\rho\pi}(\epsilon' p)(\epsilon k)]$$

$$= i\epsilon_{abc}\left[\left(\frac{m_A^2 - m_\rho^2}{2m_A}\right)^2 g_T(-\epsilon\epsilon') + \left(g_L + \frac{m_A^2 + m_\rho^2}{2m_A^2} g_T\right)(\epsilon' p)(\epsilon k)\right], \quad q = p - k,$$

and $\delta_{ab} F_A \epsilon_\lambda = \langle 0| iA_\lambda^a |A^b \epsilon p\rangle$. g_T and g_L are the $A\rho\pi$ couplings for transverse and longitudinally polarized ρ's defined by Gilman and Harari.[7]

A second sum rule can be obtained in a similar manner by interchanging the roles of ρ and A and V_λ and A_λ, dealing with the pion mass extrapolation of the matrix element $\langle \pi^a | \times V_\lambda^c | A^b\rangle$. Here we use ρ dominance of the dispersive parts in the dispersion relations and the condition $\partial^\lambda V_\lambda^c = 0$. The result is

$$f_{A\rho\pi}(0) + g_{A\rho\pi}(0) = (m_\rho^2/m_A^2)(F_A/F_\rho F_\pi). \tag{10}$$

In previous attempts[8] to apply Eq. (7), an unsubtracted dispersion relation was taken for $F_0(k^2, 0)$ rather than for $F_\pm$. A careful comparison of the two approaches shows that one cannot assume unsubtracted dispersion relations in all three amplitudes and still maintain π and A pole dominance. Since perturbation-theory models indicate that $F_\pm$ is of the order of F_0/k^2 as $k^2 \to \infty$ (indeed PCAC tells us that F_- satisfies an unsubtracted dispersion relation), we feel that the approach used in this paper is more reasonable than that adopted in Ref. 8. The same remarks apply to the derivation of the second sum rule, Eq. (10). A more complete derivation and discussion of these and other sum rules will appear elsewhere.

The results of Ref. 3 follow from Eq. (9) if $F_A(f_{A\rho\pi} - g_{A\rho\pi})$ is set equal to zero. Then $1 - 2\gamma_{\rho\pi\pi}(0,0,m_\rho^2)F_\pi^2/F_\rho = 0$ becomes Relation (1) if, by ρ dominance, we set $F_\rho = m_\rho^2/\gamma_{\rho\pi\pi}$, neglecting corrections due to taking zero pion masses in $\gamma_{\rho\pi\pi}$. There is no a priori justification for the vanishing of $F_A(f_{A\rho\pi} - g_{A\rho\pi})$, however, so that the fact that Relation (1) is approximately satisfied, combined with Eq. (9), gives the prediction

$$f_{A\rho\pi} \cong g_{A\rho\pi}, \text{ or } g_T \cong -g_L.$$

This result, by the way, is completely inconsistent with the solutions of Gilman and Harari,[7] who found $g_T = 0$, $g_L = 4/F_\pi$ by saturating Adler-Weisberger and superconvergent relations for π-ρ scattering by π, ω, and A states.

In Ref. 8 we were able to derive the result $F_A = F_\rho$; but, unfortunately we also predicted too large a value for $f_{A\rho\pi}$ ($g_{A\rho\pi}$ does not contribute to the sum rules in Ref. 8). These sum rules, therefore, correspond to a chiral-symmetry limit which, at best, is only very roughly approximated by nature. Consequently F_A must be considered an undetermined parameter in Eq. (9) and (10). We can restrict F_A, however, if we take as, a third sum rule, the one obtained by Weinberg[6] by equating the noncovariant parts of $\Delta_{\mu\nu}{}^A$ and $\Delta_{\mu\nu}{}^V$ and saturating their spectral functions by π, A, and ρ states. Rewriting it slightly, we have

$$(F_\rho{}^2/2m_\rho{}^2)(1+\xi)=F_A{}^2/m_A{}^2 \tag{11}$$

where $\xi=1-2F_\pi{}^2\gamma_{\rho\pi\pi}{}^2/m_\rho{}^2$ and $F_\rho=m_\rho{}^2/\gamma_{\rho\pi\pi}$. While Relation (1) requires ξ to be small, we would like to understand, if possible, from Eqs. (9)-(11), why this should be so. If we fix $m_A>m_\rho$ and solve for $f_{A\rho\pi}$ and $g_{A\rho\pi}$ as functions of ξ and m_A, then the character of the solutions is such that $f_{A\rho\pi}{}^2+g_{A\rho\pi}{}^2$ is minimized in the neighborhood of $\xi=0$ (i.e. the minimum is given by $\xi\cong-m_\rho{}^2/8m_A{}^2$). In other words the $A\rho\pi$ coupling is reduced considerably by virtue of Relation (1).

If we adopt Weinberg's second sum rule [Eq. (4) of Ref. 6] equating the integrals of the axial-vector and vector spectral functions of $\Delta_{\mu\nu}{}^A$ and $\Delta_{\mu\nu}{}^V$,[9] then we might expect $F_A\cong F_\rho$. However, since the integrals over the spectral functions are less rapidly convergent (indeed it is surprising that they should converge at all), the assumption of ρ and A dominance, in this case, is more questionable so that we would not be surprised if F_A deviated appreciably from F_ρ. An alternative approach[10] of treating vector and axial-vector currents as meson fields themselves can lead quite naturally to the condition $F_A=F_\rho$. Setting $F_A=F_\rho$ in Eq. (11) determines m_A in terms of ξ, $m_A{}^2=(1+\xi)^{-1}2m_\rho{}^2$. Equations (9) and (10) then fix $f_{A\rho\pi}$ and $g_{A\rho\pi}$ in terms of ξ, and, again, $f_{A\rho\pi}{}^2+g_{A\rho\pi}{}^2$ takes on its minimum values near $\xi=0$.

Finally, if we assume that the A_1 at $m_{A_1}=\sqrt{2}m_\rho$ is our A meson, then we can predict its width, using Eqs. (9)-(11), and fixing $\xi=0.13$ by using $\Gamma(\rho\to2\pi)=120$ MeV and $F_\pi=0.1\ M_p$. The result is $\Gamma(A_1\to\rho+\pi)=120$ MeV, with $F_A=1.06\ F_\rho$. The case $\xi=0$, $F_A=F_\rho$, on the other hand, predicts an A_1 width of 60 MeV. If the A_1 meson does turn out to be a bona fide resonance at this mass, then our sum rules do predict the right order of magnitude for its width.

To summarize, we have found no unambiguous derivation of Relation (1) based solely on the assumptions of PCAC, ρ dominance, and current algebra. We have obtained sum rules for the $A\rho\pi$ couplings such that Relation (1) tends to minimize these couplings. Indeed, if it were not for the attractiveness of the Weinberg sum rules,[6] we would be led to conclude from Relation (1) that there is no dominant A-meson state. It is possible that Weinberg's relations are satisfied by the continuum in the 1^+ state that contributes significantly to $\Delta_{\mu\nu}{}^A$ but not to $F_\pm$. We can only speculate, however, until the experimental situation on the existence of an A meson is clarified. In any case, Relation (1) remains of great significance to current-algebra theories of strong interactions and is, at present, somewhat of a mystery.

The author would like to thank the theoretical group at Massachusetts Institute of Technology for their helpful criticisms and for the kind hospitality they have extended to him.

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†This work is supported in part through funds provided by the U. S. Atomic Energy Commission under Contract No. AT(30-1)-2098.

*On sabbatical leave from the University of Minnesota, Minneapolis, Minnesota.

[1] $\delta_{ab}F_\pi q_\mu=\langle 0\,|\,iA_\mu{}^a\,|\,\pi^b q\rangle$. $A_\mu{}^a$ is the conventional isovector axial-vector current. The amplitude for $\pi^+\to\mu^++\nu$ is proportional to $\langle 0\,|\,A_\mu{}^1-iA_\mu{}^2\,|\,\pi^+q\rangle=-i\sqrt{2}F_\pi q_\mu$, so that this decay mode yields $F_p=0.1M_p$, M_p the proton mass. The Goldberger-Treiman formula predicts $F_\pi=(1.18M_p)/g_{\pi N}(0)$, where $g_{\pi N}(0)$ is the pion-nucleon coupling constant extrapolated by PCAC to zero pion mass. If $g_{\pi N}(0)$ is replaced by $g_{\pi N}(m_\pi{}^2)$, one finds $F_\pi=0.088M_p$. Using the latter value of F_π in expressions involving $F_\pi{}^2$ changes things by 30%. Equation (1) is satisfied best when using the larger value for F_π obtained from the charged-pion decay rate. All our states will be normalized to $(2\pi)^3 2\omega_q\delta(\vec{q}-\vec{q}\,')$.

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