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RELATIVISTICALLY INVARIANT SOLUTIONS OF CURRENT ALGEBRAS AT INFINITE MOMENTUM*

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(Received 12 July 1967)

Recently, following a suggestion by Fubini and Furlan,[1] infinite-momentum limits of weak and electromagnetic currents between one-particle states have attracted much attention. The mathematical properties of these limits have been investigated by Coester and Roepstorff.[2] According to these authors, the assumption that one-particle matrix elements of current algebras are saturated by one-particle intermediate states at infinite momentum, a suggestion particularly advocated by Dashen and Gell-Mann,[3] is incompatible with Lorentz invariance unless an infinite sequence of resonances with arbitrarily high spin values is involved. We wish to show that if such an infinite sequence of Regge-like recurrences is taken into account, there do indeed exist nontrivial solutions which are compatible with Lorentz invariance.

Matrix elements at infinite momentum.—We start with the following simple remark. Let $F^{\mu}(x)$ be a vector or axial-vector current and consider the matrix element

$$L^{\mu}=\lim_{\kappa\to\infty}\langle p_{\kappa}',N'|F^{\mu}(0)|p_{\kappa}N\rangle, \tag{1}$$

where p_{κ}' and p_{κ} denote the four-momenta of the one-particle states of mass m' and m, respectively,

$$p_{\kappa}'=(\omega_{\kappa}',\vec{p}\,'+\kappa\vec{a});\quad p_{\kappa}=(\omega_{\kappa},\vec{p}+\kappa\vec{a}).$$

Here $\vec{a}$ is a unit vector pointing in the z direction. The symbols N' and N denote the remaining quantum numbers like spin and charge. The states $|p,N\rangle$ are obtained from states at rest by means of a pure Lorentz transformation

$$|p,N\rangle=U[L(p)]|m,N\rangle(m/\omega)^{1/2},$$

where $L(p)$ is the transformation that takes the vector $m^{\mu}=(m,0,0,0)$ to the vector p^{μ}. This allows one to express the limit L^{μ} as a matrix element between states at rest:

$$L^{\mu}=\lim_{\kappa\to\infty}\left(\frac{mm'}{\omega_{\kappa}\omega_{\kappa}'}\right)^{1/2}\langle m',N'|U[L^{-1}(p_{\kappa}')L(p_{\kappa})]L_{\nu}{}^{\mu}(p_{\kappa})F^{\nu}(0)|mN\rangle.$$

The product $L^{-1}(p_{\kappa}')L(p_{\kappa})$ has a finite limit, and furthermore,

$$\lim_{\kappa\to\infty}L_{\nu}{}^{\mu}(p_{\kappa})\left(\frac{mm'}{\omega_{\kappa}\omega_{\kappa}'}\right)^{1/2}=\left(\frac{m'}{m}\right)^{1/2}a^{\mu}\bar{a}_{\nu}, \tag{2}$$

where a^{μ} and $\bar{a}^{\mu}$ denote the two lightlike vectors $a^{\mu}=(1,\vec{a})$ and $\bar{a}^{\mu}=(1,-\vec{a})$. The limit of the product $L^{-1}(p_{\kappa}')L(p_{\kappa})$ may be expressed in terms of the 2×2 representation of the homogeneous Lorentz group as follows:

$$K=\lim_{\kappa\to\infty}L^{-1}(p_{\kappa}')L(p_{\kappa})=(m'm)^{-1/2}\begin{pmatrix}m' & 0\\ -q & m\end{pmatrix}.$$

The complex number q stands for $q=q^1+iq^2$, where q^1 and q^2 are the components of the momentum transfer $\vec{q}=\vec{p}\,'-\vec{p}$ in the x-y plane. We represent the matrix K as a product

$$K=L^{-1}(\pi')QL(\pi), \tag{3}$$

where the matrices $L(\pi')$, $L(\pi)$ represent pure Lorentz transformations in the z direction that take the vectors m', m to π', π, respectively,

$$\pi^{\mu}=(2m_0)^{-1}[m_0^2+m^2,(m_0^2-m^2)\vec{a}];$$
$$\pi^{\mu\prime}=(2m_0)^{-1}[m_0^2+m'^2,(m_0^2-m'^2)\vec{a}]. \tag{4}$$

The matrix Q is given by

$$Q=\begin{pmatrix} 1 & 0 \\ -q/m_0 & 1 \end{pmatrix}. \qquad (5)$$

Note that these expressions involve the arbitrary parameter m_0. The virtue of this decomposition lies in the fact that the dependence of K on the masses m' and m is contained in the pure Lorentz transformations $L(\pi')$ and $L(\pi)$ that are independent of the momentum transfer $\vec{q}$, whereas the matrix Q is now independent of the masses. This invites us to reabsorb the transformations $U[L^{-1}(\pi)]$ and $U[L(\pi)]$ in the one-particle states with the result that

$$L^{\mu}=a^{\mu}(N'|f(\vec{q}^{\,t})|N)(2\pi)^{-3},$$
$$(N'|f(\vec{q}^{\,t})|N)=(2\pi)^3(\pi^0\pi'^0/m_0^{\,2})^{\frac{1}{2}}$$
$$\times\langle\pi'N'|U(Q)\bar{a}_{\mu}F^{\mu}(0)|\pi N\rangle. \qquad (6)$$

We have written the result in the form used by Coester and Roepstorff[2] as a matrix $(N'|f|N)$ in the space of the quantum numbers N that clearly depends only on the transverse momentum transfer $\vec{q}^{\,t}=(q^1,q^2,0)$. The above expression for L^{μ} incorporates the well-known result that the limits of the components F^1 and F^2 vanish, whereas those of F^0 and F^3 are equal. The above result is summarized in the correspondence

$$|N)\to|\pi N\rangle(2\pi)^{+3/2}(\pi^0/m_0)^{1/2},$$
$$f(\vec{q}^{\,t})\to U(Q)\bar{a}_{\mu}F^{\mu}(0). \qquad (7)$$

This correspondence makes sense only because we have eliminated the dependence of the matrix Q on the masses m' and m. We note that the finite momenta π',π point in the z direction and satisfy

$$\pi^{\mu}\bar{a}_{\mu}=\pi'^{\mu}\bar{a}_{\mu}=m_0, \qquad (8)$$

which relation determines the components of π',π in terms of the parameter m_0.

Current algebra.—Let us for simplicity focus our attention on the conserved isovector current $V_i^{\mu}(x)$, $i=1,2,3$. According to (6) the infinite momentum limit of the one-particle matrix elements of these operators are related to three operators $v_i(\vec{q}^{\,t})$ that act in the space of the quantum numbers N. The assumption that the usual local current algebra of the operators $V_i^{\mu}(x)$ when sandwiched between one-particle states at infinite momentum is saturated by one-particle intermediate states implies[2] that

$$[v_i(\vec{q}^{\,t}),v_k(\vec{q}^{\,t\prime})]=i\epsilon_{ikl}v_l(\vec{q}^{\,t}+\vec{q}^{\,t\prime}). \qquad (9)$$

Lorentz covariance.—Suppose now that a solution of this algebra, i.e., the numbers $(N'\times|v_i(\vec{q}^{\,t})|N)$, is given. Can this solution be interpreted as one-particle matrix elements of a Lorentz-covariant operator $V_i^{\mu}(0)$ according to (6)? The answer is that this is in general impossible without getting into conflict with Lorentz invariance. Let us include in the set N of quantum numbers the spin J of the particle as well as its z-component M, and let us denote the remaining quantum numbers by η. The following two conditions[4] must then be imposed on the matrix elements $(J',M',\eta'\times|v_i(\vec{q}^{\,t})|J,M,\eta)$ to guarantee that they can be interpreted as matrix elements of a covariant operator $V_i^{\mu}(0)$ according to (6):

$$(J',M',\eta'|v_i(\vec{q}^{\,t})|J,M,\eta)$$
$$=(J',M',\eta'|v_i(R_z\vec{q}^{\,t})|J,M,\eta)e^{i\theta(M'-M)}, \quad (10)$$

$$(J',\overline{M}',\eta'|v_i(\vec{q}^{\,t})|J,\overline{M},\eta)D^{J'}(\overline{R})_{M'}{}^{\overline{M}'*}D^{J}(\underline{R})^{\overline{M}}{}_{M}$$
$$=\bar{a}_{\mu}\underline{R}^{\mu}{}_{\nu}(J',M',\eta'|v_i^{\nu}(\vec{q}^{\,t})|J,M,\eta). \quad (11)$$

Here R_z denotes a rotation by angle θ around the z axis, $\overline{R}$ is a rotation around the vector $\vec{k}$ defined by $Km=(k^0,\vec{k})$ and $\underline{R}=K^{-1}\overline{R}K$. Equation (11) states that under the one-parameter group of rotations $\overline{R},\underline{R}$, the matrix element of $v_i(\vec{q}^{\,t})$ transforms like the sum of a scalar and the z-component of a vector.

Before we proceed to construct solutions that satisfy these conditions, we wish to point out that the quantities $(N'|v_i(\vec{q}^{\,t})|N)$ do not determine the operator V_i^{μ} uniquely. In fact, if V_i^{μ} is an operator whose matrix elements satisfy the relation (6), then the quantity

$$\tilde{V}_i^{\mu}=V_i^{\mu}+i[P^{\mu},G],$$

where G is an arbitrary Hermitean operator, leads to the same matrix elements. This degree of freedom is in fact a very welcome one, since it allows one to prescribe the divergence of the field $V_i^\mu(x)$ independently, for example, by specifying that $V_i^\mu(x)$ be conserved or that its divergence be dominated by the pion pole, etc.

Representations of the current algebra.—In the following we restrict ourselves to a simple, unrealistic model. More realistic solutions will be presented elsewhere. Let us suppose for simplicity that the isovector current algebra is saturated by a set of baryon states of isospin $\frac{1}{2}$ and spin values $J_1, J_2, \cdots$. In this simple case the variable η takes only two values, $\eta = \pm\frac{1}{2}$. If the solution is to be isospin invariant, we have

$$(J', M', \eta' | v_i(\vec{q}^{\,t}) | J, M, \eta) = \tfrac{1}{2}\sigma_{i\eta}^{\ \eta'}(J', M' | v(\vec{q}^{\,t}) | J, M), \qquad (12)$$

where σ_i are the Pauli matrices. This expression does indeed satisfy the current algebra (9) provided that

$$v(\vec{q}^{\,t})v(\vec{q}^{\,t'}) = v(\vec{q}^{\,t} + \vec{q}^{\,t'}), \qquad (13)$$

where $v(\vec{q}^{\,t})$ is an operator acting in the space of the quantum numbers J, M. Furthermore, in order that the operators V_i^μ be Hermitean, we must require $v(\vec{q}^{\,t})^\dagger = v(-\vec{q}^{\,t})$. We are thus looking for a unitary representation of the Abelian group in two dimensions that is compatible with the requirements of Lorentz invariance given in (10) and (11). Clearly, the one-dimensional representation[5] $v(\vec{q}^{\,t}) = \exp(i\vec{x}\cdot\vec{q}^{\,t})$ is not Lorentz invariant. In fact, the theorem by Coester and Roepstorff mentioned earlier amounts in this simple case to the statement that $v(\vec{q}^{\,t})$ must necessarily be an infinite-dimensional and therefore reducible representation of (13). A glance at the correspondence (7) invites us to interpret $v(\vec{q}^{\,t})$ as the analog of the unitary operator $U(Q)$. Note that $U(Q)$ does indeed commute with $\bar{a}_\mu V_i^\mu(0)$, which is in this case the analog of the isospin operator $\frac{1}{2}\sigma_i$. This analogy leads us to the following consideration: We note that the representation $U(Q)$ of the Abelian group of the matrices Q is in fact defined on a much larger group, the full homogeneous Lorentz group. The operator $U(Q)$ represents that particular highly reducible representation of the two-parameter Abelian group that is induced by the representation $U(\Lambda)$ of the homogeneous Lorentz group, and it is precisely this fact that is responsible for the requirements of relativistic invariance given in (10) and (11). We therefore expect to obtain a solution $v(\vec{q}^{\,t})$ that is compatible with relativistic invariance if we consider a representation $u(S)$ of the homogeneous Lorentz group or, more precisely, of its covering $S \in SL(2, C)$, and choose for $v(\vec{q}^{\,t})$ the representation induced on the subgroup of the matrices Q,

$$v(\vec{q}^{\,t}) = u(Q). \qquad (14)$$

This solution is indeed compatible with the requirement (10), because the representation $u(S)$ at the same time induces a representation of the group of rotations around the z axis with the required properties. It is also compatible with the condition (11) provided that the masses of all particles involved are the same, and we shall in the following assume that this is the case, and put $m_0 = m = m'$.

The irreducible unitary representations of the group $SL(2, C)$ are well known.[6] We restrict ourselves to the principal series of representations which are labeled by an integer m and a real parameter ρ. These representations are reducible representations of the subgroup of rotations and contain the spin values $J = \frac{1}{2}|m|, \frac{1}{2}|m| + 1, \frac{1}{2}|m| + 2, \cdots$. In order that the representation contains a particle of spin $\frac{1}{2}$, we choose $|m| = 1$ and, furthermore, put $\rho = 0$ to simplify the calculation. This representation corresponds to the infinite series of Regge-like recurrences of spin $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$, and isospin $\frac{1}{2}$. It is a straightforward matter to work out matrix elements of the type $(J', M' | u(Q) | J, M)$ for this particular representation.[7] If we restrict ourselves to $J' = J = \frac{1}{2}$, we find

$$\left(\frac{1}{2}, M' | u(Q) | \frac{1}{2}, M\right) = \begin{pmatrix} 1 & q^*/2m \\ -q/2m & 1 \end{pmatrix} \left(1 + \frac{|q|^2}{4m_0^2}\right)^{-3/2}, \qquad (15)$$

where we have collected the four possible values $M' = \pm\frac{1}{2}$, $M = \pm\frac{1}{2}$ in a 2×2 matrix. If we interpret this solution in terms of the conventional isovector form factors

$$(p', \tfrac{1}{2}, M', \eta' | V_i^\mu(0) | p, \tfrac{1}{2}, M, \eta) = \tfrac{1}{2}\sigma_{i\eta}^{\ \eta'}(2\pi)^{-3}(m^2/\omega\omega')^{\frac{1}{2}}\bar{u}(p', M')\gamma^\mu F_1^{\ v}(t) + i\sigma^{\mu\nu}q_\nu F_2^{\ v}(t)u(p, M),$$

we find that

$$F_1^v(t) = (1-t/4m^2)^{-\frac{3}{2}},$$

$$F_2^v(t) = -(1/2m)(1-t/4m^2)^{-\frac{3}{2}}. \qquad (16)$$

That the four equations for the two unknowns F_1^v and F_2^v are compatible of course reflects the Lorentz invariance of our solution. The simple solution just obtained is not expected to be realistic for at least three reasons:

(1) We do not expect that the current V_i^μ is saturated by states of isospin $\frac{1}{2}$ alone, but that matrix elements with states of isospin $\frac{3}{2}$, in particular the $(\frac{3}{2}, \frac{3}{2})$ resonance are present.

(2) The algebra of currents should probably be enlarged to include axial currents as well as currents associated with generators of SU(3).

(3) Even if we restrict ourselves to the simple model containing only states of isospin $\frac{1}{2}$, there still remains the problem of finding solutions for a nondegenerate mass spectrum. If such solutions exist, they do not show the simple dependence on the momentum transfer exhibited by (16). Note that the solution (16) corresponds to a right-hand discontinuity in the t plane of square-root character which is certainly too simple if the conventional assumptions concerning the analytic properties of form factors are correct. These points will be discussed in a forthcoming paper, together with a more detailed analysis of the role played by the group SL$(2, C)$ introduced in this Letter.[8]

We are indebted to Dr. V. Gorgé for numerous discussions and to Professor A. Mercier for reading the manuscript.

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*Work supported in part by the Swiss National Science Foundation.

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[4]The condition (10) can be found in Ref. 2, whereas (11) resembles the angular momentum condition of R. Dashen and M. Gell-Mann, Phys. Rev. Letters 17, 340 (1966). Note that an earlier version of the present paper contained the erroneous statement that the condition (10) alone guarantees Lorentz invariance.

[5]It has been shown by E. H. Roffmann, to be published, that all finite dimensional irreducible representations of the current algebra lead to a superposition of terms with a momentum dependence of this form.

[6]See, e.g., M. A. Naimark, American Mathematical Society Translations (American Mathematical Society, Providence, Rhode Island, 1957), Ser. 2, Vol. 6, p. 379.

[7]Various techniques for the evaluation of matrix elements of this kind have been proposed in connection with noncompact symmetry groups of the type SL(n, C). See, e.g., C. Fronsdal, in Proceedings of the Seminar on High Energy Physics and Elementary Particles (International Atomic Energy Agency, Vienna, Austria, 1965).

[8]After the completion of this work, we realized that solutions based on unitary representations of SL$(2, C)$ had been proposed earlier by Dashen and Gell-Mann, Ref. 4, and by S. Fubini, in Proceedings of the Fourth Coral Gables Conference on Symmetry Principles at High Energy, University of Miami, January 1967, edited by A. Perlmutter and B. Kurşunoğlu (W. H. Freeman & Company, San Francisco, California, 1967). To our knowledge, however, no explicit solutions have been presented. Finally, we wish to mention that the solution described in this Letter is closely related to a representation of the electromagnetic current proposed by A. O. Barut and Hagen Kleinert, Phys. Rev. 156, 1546 (1967), based on the use of noncompact dynamical groups.