any firm a priori conclusions about the sign of the viscous forces, although they are in the proper direction to affect the oblateness in one way or the other. Hence, these forces must be studied in more detail before any conclusions about the magnitude of the solar quadrupole moment can be drawn. The probable error associated with Einstein's prediction for the perihelion advance of Mercury might, therefore, be considerably changed.

We have indicated that photospheric magnetic fields could retard the meridional currents there. It will be interesting to see whether or not the observed oblateness fluctuates with changing magnetic activity over the present solar cycle.

The author is indebted to E. E. Salpeter, H. Y. Chiu, and R. H. Dicke for helpful criticism.

*National Academy of Sciences-National Research Council Research Associate at the Goddard Institute for Space Studies.

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GAUGE-FIELD ALGEBRA, SOFT-MESON METHODS, AND DIVERGENCES*

M. B. Halpern and G. Segrè†
Department of Physics and Lawrence Radiation Laboratory, University of California, Berkeley, California
(Received 14 July 1967)

Using Bjorken's methods in the context of the recently proposed gauge-field algebra of currents, we discuss the problem of divergences in electromagnetic mass differences ($m_{\pi^+}{}^2-m_{\pi^0}{}^2, m_{K^+}{}^2-m_{K^0}{}^2$) and decays, weak decays (e.g., $K_1{}^0\to 2\pi$), and asymptotic behavior of electromagnetic processes.

It has recently been shown by Bjorken[1] that knowledge of equal-time commutators of currents with themselves and with the Hamiltonian allows one to determine the most singular part of radiative corrections, electromagnetic mass shifts, and various other processes. He calculated the most singular parts within the context of a U(6)⊗U(6) current-algebra model. We would like to make use of some of his techniques to examine a similar class of phenomena, assuming however the equal-time commutators of the recently proposed algebra of gauge fields.[2] We assume then, with the authors of Ref. 2, the existence of a generalized Yang-Mills[3] Lagrangian coupled to pseudoscalar mesons as in the σ model[4] and, if desired, elementary fermions (as these additional particles may be introduced into the Lagrangian without altering the commutation relations). We find that all electromagnetic mass differences and decays are logarithmically divergent, and weak decays are quadratically divergent. Moreover, we will make use of partially conserved axial-vector current[5] in several calculations to see that these divergences are neglected in the limit of vanishing pseudoscalar-meson mass.

As a first example, consider the recent evaluation of the $\pi^+-\pi^0$ mass difference by Das et al.[6] They start with the standard form

$$m_{\pi^+}{}^2-m_{\pi^0}{}^2=-\frac{e^2}{4\pi}2m_\pi \operatorname{Re}\int\frac{d^4q}{q^2-i\epsilon}D_{\mu\nu}(q)\int d^4x\, e^{iq\cdot x}$$
$$\times[\langle\pi^+|T\{j_\mu{}^{\mathrm{em}}(x)j_\nu{}^{\mathrm{em}}(0)\}|\pi^+\rangle-\langle\pi^0|T\{j_\mu{}^{\mathrm{em}}(x)j_\nu{}^{\mathrm{em}}(0)\}|\pi^0\rangle], \quad (1)$$

where $j_\mu^{\text{em}}(x)$ is the electromagnetic current and $D_{\mu\nu}(q)$ is the photon propagator in the appropriate gauge for having vanishing seagull graphs.[7] Using soft-pion techniques and equal-time commutation relations[8] of the fourth components of axial-vector currents with vector currents, the matrix elements in (1) take the form of differences of vector and axial-vector current propagators. To these one can apply Weinberg's sum rules[9] for their spectral functions to eliminate the divergences and finally, in the approximation of saturating the spectral functions for vector and axial-vector current propagators by the ρ and A_1 mesons, respectively, Das et al.[6] find

$$m_{\pi^+}-m_{\pi^0}\approx 5.0 \text{ MeV},$$

in good agreement with experiment. On the other hand, if the calculation is done with finite pion mass, it is logarithmically divergent, as we proceed to discuss.

Calling p the momenta of the initial and final states, Bjorken[1] showed that the electromagnetic mass shifts are logarithmically divergent, with the coefficient of the divergence being of the form

$$M_{\mu\nu}(\gamma)$$
$$=\int d\vec{x}\langle p\gamma|[[j_\mu^{\text{em}}(\vec{x},0),H],j_\nu^{\text{em}}(0)]|p\gamma\rangle, \qquad (2)$$

where γ denotes the internal quantum numbers of the state. In the case, e.g., of the $\pi^+-\pi^0$ mass difference, one needs $M_{\mu\nu}(\pi^+)-M_{\mu\nu}(\pi^0)$; it is easily seen that only the space-space components of $M_{\mu\nu}$ need be considered. To evaluate (2), we use Eqs. (9) and (11) of Ref. 2, in the form

$$[\partial_0 j_i^{\alpha}(\vec{x},0),j_j^{\beta}(0)]=-i\left(\frac{m_0^4}{g_0^2}\right)\delta^{\alpha\beta}\delta_{ij}\delta(\vec{x})+if^{\alpha\beta\gamma}j_i^{\gamma}(\vec{x},0)\partial_j\delta(\vec{x})-i\left(\frac{g_0^2}{m_0^2}\right)f^{\alpha\delta\gamma}f^{\beta\delta\epsilon}j_i^{\gamma}(\vec{x},0)j_j^{\epsilon}(\vec{x},0)\delta(\vec{x})$$
$$+if^{\alpha\beta\gamma}\partial_i(j_j^{\gamma}(\vec{x},0)\delta(\vec{x}))+i\left(\frac{m_0^2}{g_0^2}\right)\delta^{\alpha\beta}\partial_i\partial_j\delta(\vec{x}). \qquad (3)$$

Now, whereas the U(6)⊗U(6) model gives only $\Delta I=1$ (octet) divergences, the gauge-field model also has $\Delta I=2$ (27-plet) components. Specifically, the part which contributes to the mass difference is

$$M_{lk}(\gamma)\sim(g_0^2/m_0^2)\langle p\gamma|j_l^3(0)j_k^3(0)+j_l^6(0)j_k^6(0)+j_l^7(0)j_k^7(0)|p\gamma\rangle. \qquad (4)$$

Now suppose we use soft-pion methods on this most singular part. In the limit that all pion momenta vanish, Eq. (4) goes over to

$$M_{lk}(\gamma)\to\langle 0|j_l(0)j_k(0)-j_{l5}(0)j_{k5}(0)|0\rangle, \qquad (5)$$

where the SU(3) indices are no longer relevant. Arguments are presented in Ref. 2 for setting to zero structures of this form. If we accept these arguments, we may say that the leading divergence is proportional to the meson mass and will be neglected if soft-pion methods are used.

It is also worth noting that the $\Delta I=1$ electromagnetic mass differences are, as usual,[1] logarithmically divergent. On the strength of the success of the $\pi^+-\pi^0$ mass-difference calculation, one might try to calculate the K^+-K^0 mass difference using the same techniques, that is, neglecting the logarithmic divergence proportional to m_K^2. The result is

$$m_{K^+}-m_{K^0}\simeq 1.6 \text{ MeV},$$

as compared with the experimental value of -4 MeV. Presumably this discrepancy reflects the need for a subtraction in the $\Delta I=1$ forward Compton scattering amplitude.[10,11] To summarize, the gauge-field model appears to yield logarithmically divergent $\Delta I=2$ and $\Delta I=1$ electromagnetic mass differences (baryon as well as meson) and the pseudoscalar-meson mass-difference divergences are proportional to the meson masses and are neglected in the soft-meson approximation. These conclusions apply, of course, also to $\eta\to 3\pi$ [12] in the sense that the decay rate is related to the $\pi^+-\pi^0$ and K^+-K^0 mass differences.

The next calculation we would like to discuss is a recent one[13] of $K_1^0 \to 2\pi$, assuming the weak interactions to be mediated by an intermediate vector boson. These authors use the spectral sum rules[9] and take both the pions and the kaon to be soft (or equivalently assume conservation of the axial-vector current), thus obtaining a finite result for the decay rate. The starting point of the calculation is the matrix element

$$\mathfrak{M} = \langle 2\pi | H_{\text{eff}} | K_1^0 \rangle,$$

$$H_{\text{eff}} = g^2 \int d^4y\, T\{J_\mu(y) J_\nu^+(0)\} \Delta_{\mu\nu}^B(y), \tag{6}$$

where J_μ is the weak interaction current coupled to the intermediate vector-boson B (whose propagator is $\Delta_{\mu\nu}^B$) with coupling constant g. Because the spurion term (the Hamiltonian) is explicitly included in this calculation, it will presumably not suffer from the defects of the K-mass-difference calculation. On the other hand, we can again use Bjorken's methods to isolate the leading divergence in the decay, which in this case is quadratic,[14] with a coefficient proportional to

$$\int d\vec{x} \langle 2\pi | [[j_\mu(0, \vec{x}), H], j_\nu^+(0)] | K \rangle. \tag{7}$$

Care must be exercised since the whole matrix element vanishes in the SU(3) limit; we have evaluated (7) using several generalized Yang-Mills Lagrangians with various modes of SU(3) breaking (vector-meson mass-splitting, pseudoscalar-meson octet mass splitting, or an explicit dynamical mechanism[15]). The conclusions are the same as in the mass-difference calculations: (7) does not vanish for finite π, K mass, but, in the soft π, K limit, (7) goes over into the form of Eq. (5). Hence the divergences are proportional to $m_\pi m_K$ and m_π^2. Similarly one can analyze any nonleptonic weak decay, with the conclusion that, barring unforeseen cancellations, all exhibit a quadratic divergence (proportional to meson masses if all the particles involved are pseudoscalar mesons).

Finally, we would like to mention briefly some additional consequences of the algebra of gauge fields. The first concerns the electromagnetic corrections to $\pi\beta$ decay. Bjorken[1] pointed out that the contribution of the vector part of the hadronic weak current led to a logarithmic divergence in the renormalization of the weak coupling constant and Abers, Norton, and Dicus[16] showed that Bjorken's result was in fact exact. Recently, several authors[17] have conjectured that the axial part of the hadronic weak current which depends on the commutator

$$\langle \pi^0 | [j_l^{\text{em}}(\vec{x}, 0), j_{k5}^+(0)] | \pi^- \rangle \tag{8}$$

may also be divergent, cancelling the divergence of the vector piece. Here our only observation is that in the gauge-field model the commutator (8) vanishes, and the divergence of the vector piece remains. Moreover, we note that Bjorken's[1] conjecture of a practical inequality for electron-proton scattering,

$$\lim_{q^2 \to -\infty} \lim_{E \to \infty} \frac{q^4}{\pi\alpha^2} \frac{d\sigma_p}{dq^2} \gtrsim 1 \tag{9}$$

is weakened in the gauge model, as the right-hand side of the inequality is obtained assuming the commutator of two space components of vector currents is equal to an axial current (instead of zero). Similarly, in the gauge model, we find that there is no contribution to the hyperfine structure and that the electron-positron annihilation cross section decreases faster than $1/q^4$ because the Schwinger term is finite.[18]

We would like to make a final remark, in the spirit of Ref. 17, about the possibility of finding some new algebra in which all the electromagnetic mass differences and decays, and the weak decays are finite (to lowest order in the nonstrong interactions). It appears that such an algebra would have to include

$$\int d\vec{x} [[j_\mu(\vec{x}, 0), H], j_\nu(0)] = C, \tag{10}$$

where C is a C number, and we have thus far not been able to find such a model.[19]

We would like to acknowledge very helpful discussions with Professor K. Bardacki and Professor W. Weisberger.

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*Research supported in part by the Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under Grant No. AF-AFOSR-232-66.

†Present address: Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104.

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MULTIPLE-PRODUCTION THEORY VIA TOLLER VARIABLES*

Naren F. Bali
Lawrence Radiation Laboratory, University of California, Berkeley, California

and

Geoffrey F. Chew
Lawrence Radiation Laboratory and Department of Physics, University of California, Berkeley, California

and

Alberto Pignotti†
Lawrence Radiation Laboratory, University of California, Berkeley, California
(Received 5 July 1967)

Toller's group-theoretical analysis of kinematics is exploited to define a complete set of variables, each of independent range, for particle production of arbitrary multiplicity. In terms of these variables, the generalized Regge-pole hypothesis leads to a simple, unambiguous, and experimentally accessible prediction for high-energy multiple-production cross section. A flat Pomeranchuk trajectory is shown to violate the Froissart bound.

A variety of multiperipheral models for inelastic reactions at high energy has been discussed in the literature,[1-7] but the implementing variables have been incomplete or imperfectly matched to the factorizability which characterizes such models. In this paper we exploit the work of Toller[8] to define a complete set of variables for particle production of arbitrary multiplicity, the range of each variable being independent of the others. The new variable set is natural for the implementation of any multiperipheral model, leading to a phase space that factors asymptotically in the same manner as does the amplitude. We apply our variables to the (unique) generalization of the Regge-pole hypothesis, achieving a simple, unambiguous, and experimentally accessible prediction for multiple-production cross sections at high energy which maintains the factorization property. One important aspect of the result is the exclusion of the possibility of a flat Pomeranchuk trajectory.

For the N-particle production reaction $a+b \to 1+2+\cdots+N$, we begin by selecting a particular ordering of final particles so as to define a set of $N-1$ momentum transfers Q_{mn} according to the diagram of Fig. 1. Each different ordering leads to a different set of variables; any of these sets is complete, the choice between them being a matter of convenience usu-