for pointing out this work to us after our own was completed.

[6]Mathematical aspects of such a saddle point in function space are discussed by J. S. Langer, Ann. Phys. (N.Y.) 41, 108 (1967).

[7]G. W. Rayfield and F. Reif, Phys. Rev. 136, A1194 (1964). These experiments are analogous to the inhomogeneous nucleation of liquid droplets by ions in a cloud chamber.

[8]Most of the features mentioned here can be seen explicitly in an analytic solution of the one-dimensional Landau-Ginzburg equations appropriate to superconductivity in narrow channels; see V. Ambegaokar and J. S. Langer, to be published.

[9]The mean interparticle spacing is about 3.9 Å.

[10]See B. D. Josephson, Phys. Letters 21, 608 (1966); M. E. Fisher and R. J. Burford, Phys. Rev. 156, 583 (1967); and M. E. Fisher, to be published.

[11]Since R/a enters only logarithmically, our final answer is relatively insensitive to the precise value chosen here.

[12]When the intrinsic critical velocity first sets in experimentally, the estimated radii of the critical vortices are some 10 to 50 times smaller than the nominal pore sizes of from 0.2 to 10 μ of the material in which the helium flow was observed (Ref. 1).

IMPOSSIBILITY OF BOSE CONDENSATION OR SUPERCONDUCTIVITY IN PARTIALLY FINITE GEOMETRIES

David A. Krueger
University of Washington, Seattle, Washington
(Received 26 June 1967)

The vanishing of the quasiaverages usually associated with superfluidity and superconductivity is shown for arbitrarily interacting Bose and Fermi systems which are confined to a geometry with one or more dimensions finite while one or more dimensions extend to infinity. However, it is suggested that these partially finite geometries are anomalous and are not good approximations to thin film and pore geometries found in the laboratory. The conditions on the box size which are necessary and sufficient for condensation to occur in an ideal Bose gas are also given.

Wagner[1] and Hohenberg[2] have used a general inequality obtained by Bogoliubov[3] to derive inequalities for quasiaverages in arbitrarily interacting Bose and Fermi systems. They have been applied to one- and two-dimensional systems by Hohenberg who showed that at finite temperatures the quasiaverages usually associated with the existence of superfluidity and superconductivity are zero. The purpose of this note is twofold: (a) to point out that the absence of these quasiaverages is characteristic of all Bose and Fermi systems confined to a geometry with one or more dimensions finite while one or more dimensions extend to infinity (partially finite geometries), and (b) to point out that strictly finite thin films and tubes behave more like the bulk system than like the partially finite systems.

We first consider the Bose system. Wagner's form of the Bogoliubov inequality (6.25) states

$$\langle a_{\vec{k}}^{\dagger} a_{\vec{k}}\rangle_\nu \geq \frac{mn_0\kappa T}{n\vec{k}^2+\nu m n_0^{1/2}}-\frac{1}{2}, \quad \vec{k}\neq 0, \tag{1}$$

where the quasiaverage for superfluids, $\langle a_0\rangle_\nu$, equals $n_0^{1/2}$; n_0 is the density of particles in the zero-momentum state, which is of the order of the total density n if condensation occurs, and zero otherwise; T is the temperature; κ is Boltzmann's constant; m is the Boson mass; and ν is the coefficient of the symmetry-breaking term in the Hamiltonian $\{\frac{1}{2}\nu\Omega^{1/2}(a_0+a_0^{\dagger})\}$. Wagner has discussed at length the motivation for and justification of the symmetry-breaking technique, and we will not discuss it further but will only point out one of its implications.

The density of particles is given by

$$n=\frac{\langle N\rangle}{\Omega}=\lim_{\nu\to 0}\left\{\lim_{\Omega\to\infty}\frac{1}{\Omega}\sum_{\vec{k}}\langle a_{\vec{k}}^{\dagger}a_{\vec{k}}\rangle_\nu\right\}, \tag{2}$$

where Ω is the volume of the system. Since $\langle a_{\vec{k}}^{+}a_{\vec{k}}\rangle_\nu$ is positive, we have

$$n\geq\lim_{\nu\to 0}\left\{\lim_{\Omega\to\infty}\frac{1}{\Omega}\sum{}'\langle a_{\vec{k}}^{\dagger}a_{\vec{k}}\rangle_\nu\right\}, \tag{3}$$

where the prime indicates a summation over $\{\vec{k}\}$, where $\{\vec{k}\}$ is a subset of the allowed $\vec{k}$ values. Since periodic boundary conditions have been used in the proof of Wagner's inequality, the allowed values of $\vec{k}$ for a box geometry $(L_1\times L_2\times L_3)$ are

$$\vec{k}=2\pi\left(\frac{l_1}{L_1},\frac{l_2}{L_2},\frac{l_3}{L_3}\right);\ l_i=0,\pm1,\pm2,\cdots. \tag{4}$$

First consider a geometry with L_1 extending to infinity but with L_2 and L_3 finite. The proof by contradiction proceeds by first assuming $n_0>0$. Next, taking $\{\vec{k}\}$ such that $l_2=0=l_3$ and $1\leq l_1\leq l_{\max}$ and using Wagner's inequality, we have

$$n\geq\lim_{\nu\to0}\left\{\lim_{L_1\to\infty}\frac{1}{L_1L_2L_3}\sum_{l_1=1}^{l_{\max}}\left[\frac{mn_0\kappa T}{4\pi^2nl_1^2/L_1^2+\nu m\sqrt{n_0}}-\frac{1}{2}\right]\right\}. \tag{5}$$

Taking

$$l_{\max}=\frac{L_1}{2\pi}\left[\frac{m}{n}(2n_0\kappa T-\nu\sqrt{n_0})\right]^{1/2},$$

the summand is always positive and the $(-\frac{1}{2})$ term yields a finite negative number. However, we now show that the first term goes to infinity for finite n_0. Noting that

$$\sum_{l=1}^{l_{\max}}\frac{1}{l^2+C^2}\geq\int_1^{l_{\max}}\frac{dl}{l^2+C^2}=\frac{1}{C}\left\{\tan^{-1}\left(\frac{l_{\max}}{C}\right)-\tan^{-1}\left(\frac{1}{C}\right)\right\}, \tag{6}$$

we see that

$$n\geq\lim_{\nu\to0}\left\{\frac{\kappa T}{2\pi L_2L_3}\left(\frac{mn_0^{3/2}}{n\nu}\right)^{1/2}\tan^{-1}\left(\frac{2\kappa T\sqrt{n_0}}{\nu}-1\right)\right\}-\frac{1}{4\pi L_2L_3}\left\{\frac{2n_0\kappa Tm}{n}\right\}^{1/2}, \tag{7}$$

which implies that n_0 must be zero for n to be finite at finite temperatures.

A similar proof goes through for the case where L_1 and L_2 tend to infinity with L_3 finite:

$$n\geq\lim_{\nu\to0}\left\{\lim_{L_1L_2\to\infty}\frac{1}{L_1L_2L_3}\sum{}'\left[\frac{mn_0\kappa T}{4\pi^2n(l_1^2/L_1^2+l_2^2/L_2^2)+m\nu\sqrt{n_0}}-\frac{1}{2}\right]\right\},$$

$$n\geq\lim_{\nu\to0}\frac{\pi}{2}\int_0^{R_{\max}}rdr\left[\frac{mn_0\kappa T}{4\pi^2nr^2+m\nu\sqrt{n_0}}-\frac{1}{2}\right].$$

Taking $R_{\max}^2=(2n_0\kappa Tm-\nu m\sqrt{n_0})/4\pi^2n$, we find that

$$n\geq\lim_{\nu\to0}\left\{\frac{mn_0\kappa T}{16\pi L_3n}\ln\left[\frac{2\kappa T\sqrt{n_0}}{\nu}\right]-\frac{mn_0\kappa T}{16\pi nL_3}\right\}, \tag{8}$$

which again shows that for finite n we must have $n_0=0$ at finite temperatures. For the free particle case, the result that $n_0=0$ agrees with that obtained by Osborne[4] and Ziman[5] who investigated this problem without using a symmetry-breaking term in the Hamiltonian.

A similar result for fermions in partially-finite geometries follows from Hohenberg's equations (32) and (33):

$$F(\vec{k})\geq(2mT\kappa/n\vec{k}^2)|\Delta+\eta(\vec{k})|^2-R(\vec{k}), \tag{9}$$

$$\frac{1}{\Omega}\sum_{\vec{k}}F(\vec{k})<\infty, \tag{10}$$

where Δ is a quasiaverage, $\eta(\vec{k})$ is regular for small $\vec{k}$ and approaches Δ as $\vec{k}$ goes to zero, and $R(\vec{k})$ is regular for small $\vec{k}$. Using the same technique as before, we see that in partially-finite geometries must be zero.

One might question these results because the proofs depend crucially on taking $l_3=0$, which is allowed for periodic boundary conditions but is not allowed for box boundary conditions. We can only point out that the basic inequalities have been proved only for periodic boundary conditions. Explicit calculations for the free-particle Bose gas show that the result is, in fact, true for either box or periodic boundary conditions.

If one accepts the existence of quasiaverages as the origin of "superproperties," the above results seem to contradict the experimental evidence that liquid helium behaves as a superfluid even when confined to pores or thin films and that thin-film superconductors do exist. A simple explanation is suggested if we note that condensation is always possible in a free-particle Bose gas confined to a strictly finite geometry but is not possible in partially finite geometries. Explicit calculations for an ideal Bose gas show that the necessary and sufficient conditions for condensation (i.e., n_0 is of order N and increases linearly with N) are

$$L_1 \geq L_2 \geq L_3 \gg \lambda_T \quad T < T_c,$$

$$L_3 \geq \frac{1}{1-(T/T_c)^{3/2}} \frac{1}{\pi[F_{\frac{3}{2}}(0)]^{2/3}} \times n^{-1/3} \frac{T}{T_c} \left\{ \frac{L_1}{L_2} + 2\pi \ln \frac{L_2}{L_3} \right\}, \quad (11)$$

$$L_1 \geq L_2 \gg \lambda_T \gtrsim L_3,$$

$$L_3 \geq \frac{1}{\pi[F_{\frac{3}{2}}(0)]^{2/3}} n^{-1/3} \frac{T}{T_c} \left\{ \frac{L_1}{L_2} + 2\pi \ln \frac{L_2}{\lambda_T} \right\}, \quad (12)$$

and for $L_1 \gg \lambda_T \gtrsim L_2 \gtrsim L_3$,

$$L_3 \geq \frac{\pi}{[F_{\frac{3}{2}}(0)]^{2/3}} n^{-1/3} \frac{T}{T_c} \frac{L_1}{L_2}, \quad (13)$$

where $F_{3/2}(0) = 2.612$ and

$$T_c = \frac{2\pi\hbar^2}{m\kappa} \left\{ \frac{n}{F_{3/2}(0)} \right\}^{2/3},$$

the ideal bulk critical temperature. For $L_1 = L_2 \approx 1$ cm and $T \approx \frac{1}{2} T_c$, Eq. (11) requires that $L_3 \gtrsim 45$ Å for condensation to occur.[6] This implies that partially finite geometry models are anomalous and will not be good approximations to films or pore geometries in the laboratory. De Wames, Lehman, and Wolfram[7] have made a similar observation with respect to letting the dimensions go to infinity in one- and two-dimensional superconductors. We should emphasize, however, that the present result is valid for all partially finite systems (e.g., slabs and tubes of arbitrary but finite thickness). This is not to say that the usual methods for the bulk system fail. In fact, it appears that for bosons in strictly finite geometries, the "bulk-type" expression

$$\frac{1}{\Omega} \sum_k f(\vec{k}) \simeq \frac{f(0)}{\Omega} + \frac{1}{D^{3-\eta}} \int \frac{d^\eta k}{(2\pi)^\eta} f(\vec{k}), \quad (14)$$

where $\Omega + L^\eta D^{3-\eta}$ and $\infty > L \gg \lambda_T \gtrsim D$ with λ_T the thermal Debroglie wavelength) gives a physically more reasonable result than the mathematically exact partially finite geometry calculation, even for film and pore geometries. This conjecture has been verified mathematically for the free-particle Bose gas where we have shown that the corrections to the right-hand side of Eq. (14) are very small [less than or of order $\lambda_T/L \ll 1$ in the calculation of $\lambda_T^3(N/\Omega) \simeq 1$] for $\eta = 1$, 2, and 3.

A final point concerns the connection between off-diagonal long-range order (ODLRO) and a finite value of n_0 for Bose systems. Girardeau[8] has shown that ODLRO is possible in the presence of a generalized Bose condensation which does not require a finite n_0. Thus it appears that the existence of ODLRO for bosons in partially finite geometries is still an open question.

The author has benefited from discussions with Dr. R. Puff.

---

[1]H. Wagner, Z. Physik 195, 273 (1966).

[2]P. C. Hohenberg, Phys. Rev. 158, 383 (1967).

[3]N. N. Bogoliubov, Phys. Abhandl. Sowjetunion 6, 113, 229 (1962).

[4]M. F. M. Osborne, Phys. Rev. 76, 396 (1949).

[5]J. M. Ziman, Phil. Mag. 44, 548 (1953).

[6]After this work had been submitted for publication, the author learned of similar recent unpublished results by G. V. Chester which use the Bogoliubov inequality directly for the interacting system.

[7]R. E. De Wames, G. W. Lehman, and T. Wolfram, Phys. Rev. Letters 13, 749 (1964).

[8]M. D. Girardeau, J. Math. Phys. 6, 1083 (1965).