INTRINSIC CRITICAL VELOCITY OF A SUPERFLUID*

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(Received 30 June 1967)

A fluctuation theory of the intrinsic critical velocity of a superfluid near the lambda point is developed in analogy with the droplet model of condensation of a supersaturated vapor. For the superfluid, vortex rings play the role of critical droplets in homogeneously nucleating transitions to states of lower superflow. A quantitative estimate is in reasonable agreement with recent experiments on helium.

Recent experiments by Clow and Reppy[1] on the superflow of helium near the lambda point seem to imply the existence of a critical velocity essentially independent of the shape and dimensions of the container.[2] This intrinsic critical velocity varies approximately[1] as

$$v_c(T) \approx u_c[1-(T/T_\lambda)]^{2/3} \quad (T \to T_\lambda), \tag{1}$$

where $u_c \simeq 380$ cm/sec. We will present a theoretical argument indicating that $v_c(T)$ is proportional to the superfluid density, $\rho_s(T)$. In combination with the observed result[3]

$$\rho_s(T)/\rho \approx A[1-(T/T_\lambda)]^{2/3} \quad (T \to T_\lambda) \tag{2}$$

with $A \simeq 2.40$, our arguments yield (1) with a constant u_c of correct order of magnitude.

Our basic assertion is that a nonzero superfluid flow must be regarded as a metastable state with properties analogous to those of a supersaturated vapor. For a given macroscopic sample of the metastable phase, there is a finite probability per unit time for homogeneous nucleation of the stable phase. But for a sufficiently small superflow (or supersaturation), this probability rate is too small to be observable; so the system appears stable. Conversely the critical superflow (or supersaturation) is achieved when the probability of nucleation becomes appreciable within experimental times. Such an analogy has been sketched previously by Vinen[4] and developed in greater detail by Iordanskii.[5] Our approach and analysis is close to Iordanskii's in important respects but differs in other significant ways.

To discuss superfluid helium we assume that the relevant states of the system can be described by a complex-values quasilocal continuous order parameter $\psi(\vec{r})$. The a priori probability that the system will be found in a state $\psi(\vec{r})$ is proportional to a Boltzmann factor $\exp[-F\{\psi(\vec{r})\}/k_BT]$. The various stable and metastable states correspond to local minima of the effective-free-energy functional $F\{\psi(\vec{r})\}$. The statistical fluctuations appropriate to an isothermal canonical ensemble will be visualized as a continuous random motion of the system point ψ through the function space, the vicinity of any point being visited with a frequency proportional to the Boltzmann factor.

Consider the superfluid in a state close to some ψ_m locating a relative but not absolute minimum of $F\{\psi\}$. To pass continuously from ψ_m to a neighboring minimum, say ψ_m', of lower free energy, the system point must move through regions of higher free energy and lower statistical weight. The least improbable fluctuation which can carry the system from ψ_m to $\psi_{m'}$ corresponds to the lowest saddle point of $F\{\psi\}$ on paths from ψ_m to $\psi_{m'}$. The "barrier" height ΔF of this saddle point, relative to the minimum at ψ_m, determines the rate at which the metastable state decays. On general grounds the lowest saddle point is expected[6] to describe a state $\psi_c(\vec{r})$ which is close to the metastable state $\psi_m(\vec{r})$ almost everywhere but which contains a single localized fluctuation. For a supersaturated vapor this fluctuation represents a liquid droplet just large enough to nucleate condensation, but for a superfluid we argue that it must be a vortex ring.

In the first place, the saddle-point criterion requires that $\psi_c(\vec{r})$ be a stationary point of $F\{\psi\}$:

$$[\delta/\delta\psi(\vec{r})]F\{\psi(\vec{r})\}_{\psi=\psi_c} = 0. \tag{3}$$

This equation is just the Landau-Ginzburg equation (or its generalization) which is known to describe a current-conserving superfluid hydrodynamics in which vorticity can occur but is quantized. Vortex rings consistent with these expectations have been observed by Rayfield and Reif.[7]

Secondly, note that a vortex ring has just the topological properties needed to nucleate a transition from a state, say $\psi_{\vec{k}}(\vec{r}) = A\exp(i\vec{k}\cdot\vec{r})$, of uniform superflow at velocity $v_s = (\hbar/m)|\vec{k}|$, to a similar state $\psi_{\vec{k}'}(\vec{r})$ with one less wavelength in the length L of the (periodic) container and correspondingly lower superfluid velocity $v_s' = (\hbar/m)(k-2\pi/L)$. This is illustrated schematically in Fig. 1. Initially (a) the magnitude of ψ decreases locally within a (relatively small) region. Next, (b) $|\psi|$ vanishes identically at one point. Such a zero of $|\psi|$ is a topological necessity if an over-all change of phase difference is to be accomplished continuously. The probable form (c) of the critical saddle point $\psi_c(\vec{r})$ is a ring of unit vorticity for which the change of phase passing around the container through the center of the ring is 2π less than the phase change obtained along any similar path passing outside the ring.[8] Finally, in (d) and (e) the vortex ring expands, lowering the free energy again. If it is subsequently annihilated at the walls, the order parameter ψ loses one wavelength across the whole system.

For an initial quantitative estimate of the nucleation rate we assume that vortex rings near critical size may be described by the classical hydrodynamical formulas. Although tested by Rayfield and Reif[7] for $T \simeq 0.2T_\lambda$, we expect these expressions to yield an overestimate of ΔF and hence of the critical velocity, since they assume incompressible flow outside a small vortex core and thus neglect variations of the amplitude $|\psi|$ which may well be important near T_λ.

Suppose, as in Fig. 1, that the superfluid flows to the right with velocity v_s, and consider a vortex ring of radius R. Since Eq. (3) is time independent, the critical ring must be stationary in the laboratory frame; this result, however, also follows from free-energy considerations. In the rest frame of the fluid (denoted by subscript zeros) the energy E_0 and velocity v_0 of a classical vortex ring are determined by the radius:

$$E_0 = \tfrac{1}{2}\rho_s \kappa^2 R(\eta - \tfrac{7}{4}), \quad \eta = \ln(8R/a), \tag{4}$$

$$v_0 = \kappa(\eta - \tfrac{1}{4})/4\pi R, \tag{5}$$

where $\kappa = (h/m)$ is the vorticity, and a is the radius of the vortex core assumed to satisfy $(a/R)^2 \ll 1$. If the ring moves to the left in the rest frame, the energy in the laboratory frame is

$$E = E_0 - p_0 v_s, \tag{6}$$

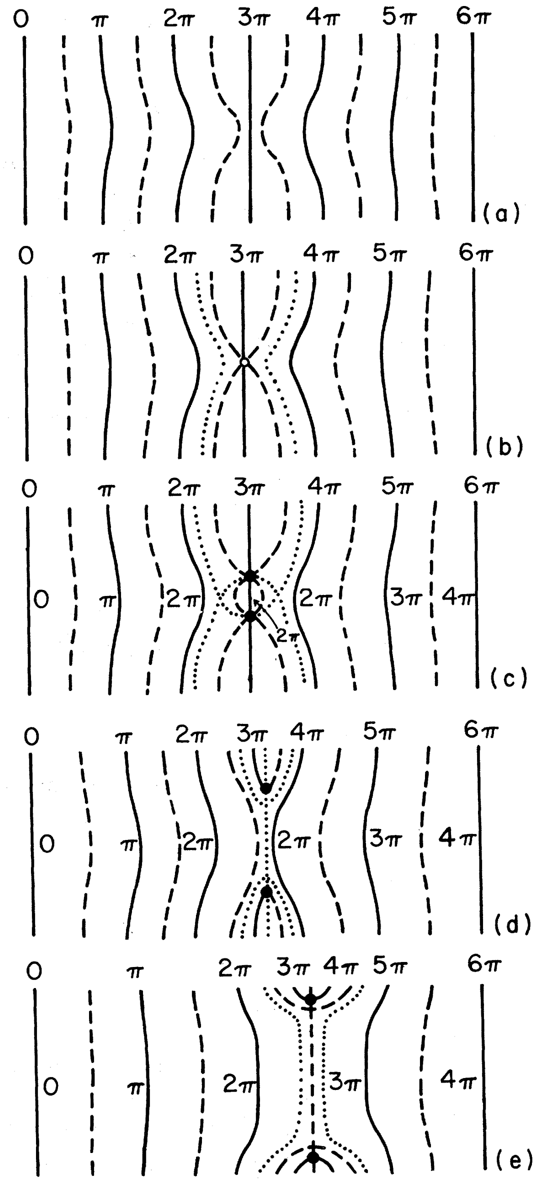

FIG. 1. Cross sections of the surfaces of constant phase $\varphi = \arg\psi(\vec{r})$ illustrating schematically the successive stages in the nucleation of a critical vortex ring (c) and its subsequent expansion. Surfaces for $\varphi = n\pi$ and $(n+\frac{1}{2})\pi$ are shown as solid lines and dashed lines, respectively; the extra surfaces, shown as dots reveal further detail; open circles denote the first vanishing of $|\psi|$ and the centers of the vortex cores. [Cylindrical symmetry about a central axis is presumed.] The labeling indicates the different total phase changes obtained on passing through or outside the ring.

where the momentum p_0 is determined by

$$v_0 = dE_0/dp_0 \equiv (dE_0/dR)(dp_0/dR)^{-1}. \tag{7}$$

From (7) one finds $p_0 \propto R^2$ in accord with classical theory. Thus $E=E(R)$ has a maximum at $R=R_c$ determined by

$$(dE_0/dR)_{R=R_c} - v_s(dp_0/dR)_{R=R_c} = 0. \quad (8)$$

Evidently $E_c = E(R_c)$ is the critical barrier height ΔF. Vortex rings larger than R_c lower their free energy by expanding (and slowing down) while smaller rings tend to collapse. Comparison with Eq. (7) shows that (8) is solved by $v_0 = v_0(R_c) = v_s$, which proves our assertion that the critical fluctuation is stationary in the laboratory frame.

Our final estimate for the free energy of the critical fluctuation is

$$\Delta F \simeq E_c \simeq \frac{\rho_s \kappa^3}{16\pi v_s}(\eta - \tfrac{1}{4})(\eta - 11/4), \quad (9)$$

valid to leading order in $(a/R)^2$. To deduce a critical velocity from this, recall that the nucleation rate per unit volume, f, must be of the form

$$f \approx f_0 \exp(-\Delta F/k_B T), \quad (10)$$

where f_0 is the characteristic rate for microscopic processes. If f exceeds some minimum observable laboratory frequency, f_L, the superfluid velocity v_s must exceed a critical velocity determined from (9) and (10) as

$$v_c \simeq \frac{\rho_s(T)h^3(\eta-\frac{1}{4})(\eta-\frac{11}{4})}{16\pi m^3 k_B T \ln(f_0/f_L)}. \quad (11)$$

Now the characteristic frequencies of processes occurring on an atomic scale are of order 5×10^{12} sec^{-1} as determined, for example, by the speed of sound and the atomic diameter. Since the number density of helium at T_λ is 2×10^{22} cm^{-3}, we obtain the estimate $f_0 \simeq 10^{35}$ cm^{-3} sec^{-1}. It is reasonable to take f_L as, say, 0.1 cm^{-3} sec^{-1}, which gives $\ln(f_0/f_L) \simeq 83$. Note that an error of a few factors of 10 in f_0 or f_L would alter the calculated v_c by only a few percent!

Near T_λ we may combine (11) with (2) to obtain (1) with the constant

$$u_c = \frac{A\rho_\lambda h^3(\eta-\frac{1}{4})(\eta-\frac{11}{4})}{16\pi m^3 k_B T_\lambda \ln(f_0/f_L)}$$

$$\simeq 224(\eta-\tfrac{1}{4})(\eta-\tfrac{11}{4}) \text{ cm sec}^{-1}. \quad (12)$$

Now $\eta = \ln(8R_c/a)$ depends implicitly on v_c through (5), which may be rewritten for the radius of the critical vortex as

$$R_c = (\eta-\tfrac{1}{4})\Lambda_c/4\pi$$

$$\simeq 33(\eta-\tfrac{11}{4})^{-1}[1-(T/T_\lambda)]^{-2/3} \text{ Å}, \quad (13)$$

where $\Lambda_c = 2\pi/k_c$ is the wavelength appropriate to the critical flow velocity. As might have been guessed, this is also roughly the magnitude of R_c. Far below T_c one may suppose that the core radius is $a \simeq 4$ Å in accord with the atomic dimensions.[9] For deviations $(T_\lambda - T)$ of from 1 to 0.1 % of T_λ, this leads to $\eta \simeq 6.1$ to 7.3, and thence $u_c \simeq (4\text{-}7)\times10^3$ cm sec^{-1}. However, near T_λ we should rather expect that the core radius will be approximately equal to the correlation length $\xi(T)$. This we expect, by comparison with critical phenomena in other systems,[10] to vary as

$$\xi \simeq \xi_0[1-(T/T_\lambda)]^{-\nu}, \quad (14)$$

with $\nu \simeq \frac{2}{3}$ and $\xi_0 \simeq 2$ Å. With this estimate for a, the ratio R/a should be roughly constant near T_λ with a value of about 10.[11] This finally yields

$$u_c \lesssim 1500 \text{ cm sec}^{-1}, \quad (15)$$

the inequality serving to recall that, for the reasons explained, we still expect this value to be an overestimate. In fact, it is about 4 times the experimentally observed value.[12]

We should like to thank Professor J. D. Reppy, Professor G. V. Chester, and Professor W. F. Vinen for informative discussions.

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*Work supported in part by the National Science Foundation and the Advanced Research Projects Agency through the Materials Science Center at Cornell University, Ithaca, New York.

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[1]J. R. Clow and J. D. Reppy, Phys. Rev. Letters 19, 289 (1967).

[2]In contrast to previously observed critical velocities, see W. F. Vinen, in Liquid Helium, Proceedings of the Enrico Fermi International School of Physics, Course XXI, edited by G. Careri (Academic Press, Inc., New York, 1963), p. 336.

[3]J. R. Clow and J. D. Reppy, Phys. Rev. Letters 16, 887 (1966); J. A. Tyson and D. H. Douglass, Phys. Rev. Letters 17, 472, 622 (1966).

[4]W. F. Vinen, loc. cit., Sec. 3.3.

[5]S. V. Iordanskii, Zh. Eksperim. i Teor. Fiz. 48, 708 (1965) [translation: Soviet Phys.—JETP 21, 467 (1965)]. We are indebted to Professor Vinen

for pointing out this work to us after our own was completed.

[6]Mathematical aspects of such a saddle point in function space are discussed by J. S. Langer, Ann. Phys. (N.Y.) 41, 108 (1967).

[7]G. W. Rayfield and F. Reif, Phys. Rev. 136, A1194 (1964). These experiments are analogous to the inhomogeneous nucleation of liquid droplets by ions in a cloud chamber.

[8]Most of the features mentioned here can be seen explicitly in an analytic solution of the one-dimensional Landau-Ginzburg equations appropriate to superconductivity in narrow channels; see V. Ambegaokar and J. S. Langer, to be published.

[9]The mean interparticle spacing is about 3.9 Å.

[10]See B. D. Josephson, Phys. Letters 21, 608 (1966); M. E. Fisher and R. J. Burford, Phys. Rev. 156, 583 (1967); and M. E. Fisher, to be published.

[11]Since R/a enters only logarithmically, our final answer is relatively insensitive to the precise value chosen here.

[12]When the intrinsic critical velocity first sets in experimentally, the estimated radii of the critical vortices are some 10 to 50 times smaller than the nominal pore sizes of from 0.2 to 10 μ of the material in which the helium flow was observed (Ref. 1).

IMPOSSIBILITY OF BOSE CONDENSATION OR SUPERCONDUCTIVITY IN PARTIALLY FINITE GEOMETRIES

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(Received 26 June 1967)

The vanishing of the quasiaverages usually associated with superfluidity and superconductivity is shown for arbitrarily interacting Bose and Fermi systems which are confined to a geometry with one or more dimensions finite while one or more dimensions extend to infinity. However, it is suggested that these partially finite geometries are anomalous and are not good approximations to thin film and pore geometries found in the laboratory. The conditions on the box size which are necessary and sufficient for condensation to occur in an ideal Bose gas are also given.

Wagner[1] and Hohenberg[2] have used a general inequality obtained by Bogoliubov[3] to derive inequalities for quasiaverages in arbitrarily interacting Bose and Fermi systems. They have been applied to one- and two-dimensional systems by Hohenberg who showed that at finite temperatures the quasiaverages usually associated with the existence of superfluidity and superconductivity are zero. The purpose of this note is twofold: (a) to point out that the absence of these quasiaverages is characteristic of all Bose and Fermi systems confined to a geometry with one or more dimensions finite while one or more dimensions extend to infinity (partially finite geometries), and (b) to point out that strictly finite thin films and tubes behave more like the bulk system than like the partially finite systems.

We first consider the Bose system. Wagner's form of the Bogoliubov inequality (6.25) states

$$\langle a_{\vec{k}}^\dagger a_{\vec{k}}\rangle_\nu \geq \frac{mn_0\kappa T}{n\vec{k}^2+\nu m n_0^{1/2}}-\frac{1}{2},\quad \vec{k}\neq 0, \tag{1}$$

where the quasiaverage for superfluids, $\langle a_0\rangle_\nu$, equals $n_0^{1/2}$; n_0 is the density of particles in the zero-momentum state, which is of the order of the total density n if condensation occurs, and zero otherwise; T is the temperature; κ is Boltzmann's constant; m is the Boson mass; and ν is the coefficient of the symmetry-breaking term in the Hamiltonian $\{\frac{1}{2}\nu\Omega^{1/2}(a_0+a_0^\dagger)\}$. Wagner has discussed at length the motivation for and justification of the symmetry-breaking technique, and we will not discuss it further but will only point out one of its implications.

The density of particles is given by

$$n=\frac{\langle N\rangle}{\Omega}=\lim_{\nu\to 0}\left\{\lim_{\Omega\to\infty}\frac{1}{\Omega}\sum_{\vec{k}}\langle a_{\vec{k}}^\dagger a_{\vec{k}}\rangle_\nu\right\}, \tag{2}$$

where Ω is the volume of the system. Since $\langle a_{\vec{k}}^\dagger a_{\vec{k}}\rangle_\nu$ is positive, we have

$$n\geq\lim_{\nu\to 0}\left\{\lim_{\Omega\to\infty}\frac{1}{\Omega}\sum{}'\langle a_{\vec{k}}^\dagger a_{\vec{k}}\rangle_\nu\right\}, \tag{3}$$

where the prime indicates a summation over $\{\vec{k}\}$, where $\{\vec{k}\}$ is a subset of the allowed $\vec{k}$ values. Since periodic boundary conditions have been used in the proof of Wagner's inequality,