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KOHN VARIATIONAL PRINCIPLE FOR THREE-PARTICLE SCATTERING*

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The Kohn variational principle is extended to apply to scattering processes where a two-particle bound state is broken up by a third particle.

In the past it has often proved that use of a variational principle[1] has been the most efficient method of making quantitative calculations of the properties of systems of several particles (more than two). In particular, the Kohn-Hulthén principle has been applied successfully to problems where the open channels contain only two bound parts, as for example, electron-hydrogen atom scattering.[2] In this Letter, we discuss how the principle may be extended to include open channels containing three separated particles.

As an example, we take the case of three different particles of equal mass ($m = \frac{1}{2}$) interacting through two-body potentials that are superpositions of Yukawa potentials. We assume that each pair may form a number of bound states labeled by α_i with binding energies $E_\alpha{}^i$ ($i = 1, 2, 3$ denotes the particle that is not bound). In practice the initial state of the system will contain one of these bound pairs, say 2 and 3, and will be described by the state $|\chi\rangle = |\chi_1{}^1, \vec{p}'\rangle$, an eigenstate of $H_1 = H_0 + V_1$, energy E, where H_0 is the kinetic energy of all three particles and V_1 the interaction between 2 and 3. We work in the frame with total momentum zero, and $\vec{p}'$ is the initial momentum of particle 1.

Using Eq. (184), p. 102, of Goldberger and Watson[3] we may write the scattering wave function corresponding to initial state $|\chi\rangle$ as

$$\psi^+(\hat{\rho}_i) = \chi(\hat{\rho}_i) + (2\pi)^{-9/2}3^{-3/2}\int d\hat{K}_i \exp(i\hat{\rho}_i\cdot\hat{K}_i)(E-\hat{K}^2+i\epsilon)^{-1}\langle\hat{K}_i|T^+(E)|\chi\rangle, \tag{1}$$

where we have used a notation similar to Lovelace[4] such that $\hat{\rho}_i$ is a six-dimensional vector

$$\hat{\rho}_1 = (\vec{X}_1, \vec{Y}_1) = \{(\tfrac{2}{3})^{1/2}[\vec{r}_1 - \tfrac{1}{2}(\vec{r}_2+\vec{r}_3)], (2)^{-1/2}[\vec{r}_2-\vec{r}_3]\} \tag{2}$$

and

$$\hat{K}_1 = (\vec{P}_1, \vec{Q}_1) = \{(\tfrac{2}{3})^{1/2}\vec{p}_1, (2)^{-1/2}[\vec{p}_2-\vec{p}_3]\} \tag{3}$$

and similarly for $i = 2, 3$.

Alternatively, proceeding from Eq. (81), p. 79 of Ref. 3, taking the $\chi_b{}^i$ as eigenstates of $H_i = H_0 + V_i$, we find

$$\psi^+(\hat{\rho}_i) = \chi(\hat{\rho}_i) + (2\pi)^{-3}3^{-\frac{3}{2}}\int d\hat{K}_i \psi_{\vec{Q}_i}{}^-(\vec{Y}_i)\exp(i\vec{X}_i\cdot\vec{P}_i)(E-\hat{K}^2+i\epsilon)^{-1}\langle\hat{K}_i|\overline{T}_i{}^+|\chi\rangle$$
$$+(2\pi)^{-3}(\tfrac{2}{3})^{\frac{3}{2}}\sum_{\alpha_i}\int d\vec{P}_i\chi_\alpha{}^i(\vec{Y}_i)\exp(i\vec{X}_i\cdot\vec{P}_i)(E+E_\alpha{}^i-\vec{p}_i{}^2+i\epsilon)^{-1}\langle\chi_\alpha{}^i, \vec{P}_i|T(E)|\chi\rangle. \tag{4}$$

The matrix elements $\langle \hat{K}_i | T^+(E) | \chi\rangle$ and $\langle \chi_\alpha{}^i, \vec{P}_i | T^+(E) | \chi\rangle$ are the usual reduced T-matrix elements for scattering from the initial state to a three-particle state and a bound pair, respectively, with the final states off the energy shell. However, $\langle \hat{K}_i | \overline{T}^+ | \chi\rangle$ is given by

$$\langle \hat{K}_i | \overline{T}_i{}^+ | \chi\rangle = \langle \psi_{\vec{Q}_i}{}^-, \vec{P}_i | [1 + V^i(E-H+i\epsilon)^{-1}] V^1 | \chi\rangle$$
$$= \langle \hat{K}_i | [1 + V_i(\hat{K}^2 - H_i + i\epsilon)^{-1}][1 + V^i(E-H+i\epsilon)^{-1}] V^1 | \chi\rangle \quad (5)$$

which is equal to $\langle \hat{K} | T^+(E) | \chi\rangle$ on the energy shell $\hat{K}^2 = E$. We have used $V^i = V - V_i$.

In Eq. (4), $\psi_{\vec{Q}_i}{}^-(\vec{Y})$ and $\chi_\alpha{}^i(\vec{Y}_i)$ are scattering and bound-state wave functions of the relative coordinate $\vec{Y}_i$, and

$$\chi(\hat{\rho}_1) = (2\pi)^{-3} \chi_1{}^1(\vec{Y}_1) e^{i\vec{X}_1 \cdot \vec{P}'}. \quad (6)$$

We stress that Eqs. (1) and (4) hold whichever index i is placed on $\hat{\rho}, \hat{K}$, etc.

For the variational principle, we need to know the asymptotic form of $\psi^+(\rho)$ for large ρ. Equations (1) and (4) provide this information if we make use of the analytic properties of the T-matrix elements appearing there. Using the methods of Rubin, Sugar, and Tiktopoulos,[5] we may deduce that, near real $\hat{K}$, the only singularities of $\langle \hat{K} | T^+(E) | \chi\rangle$ are subenergy bound-state poles and normal thresholds occurring at

$$\vec{P}_j{}^2 = (E + E_\alpha{}^j + i\epsilon), \quad \alpha = 1, 2, \cdots$$

and

$$\vec{P}_j{}^2 = (E + i\epsilon) \quad (7)$$

for all j. The singularities of $\langle \hat{K}_i | \overline{T}_i{}^+ | \chi\rangle$ are the same except that for $j = i$, (7) is replaced by $\vec{Q}_i{}^2 = 0$. At the singularity $\vec{Q}^2 = 0$, the on-shell matrix element $\langle \hat{K} | T^+(E) | \chi\rangle$ is actually an analytic function of $Q_i = (\vec{Q}_i{}^2)^{1/2}$ (see Eden et al.,[6] p. 230).

Let us suppose that $\hat{\rho} = \rho\hat{\rho}_u$, with $\hat{\rho}_u$ a unit six-vector and ρ large, and slightly distort the integration contour in (1), to $\hat{K} \to \hat{K} + i\delta\hat{K}$. We would like to choose the distortion so that $\mathrm{Re}(i\hat{\rho} \cdot \hat{K}) < 0$ and also so that the singularities of $(E - \hat{K}^2 + i\epsilon)^{-1}$ and $\langle \hat{K} | T^+(E) | \chi\rangle$ are not crossed. Providing $\hat{\rho}_u$ is chosen so that $\vec{Y}_i \neq 0$ (all i), we may show that this is possible except near $\hat{K} = E^{1/2}\hat{\rho}_u$. Near here $\langle \hat{K} | T^+(E) | \chi\rangle$ is analytic, and we may replace it by $\langle E^{1/2}\hat{\rho}_u | T^+(E) | \chi\rangle$ according to the usual method of stationary phase. Consequently, using the Appendix of Schwartz and Zemach,[7] we find that

$$\psi^\pm(\hat{\rho}) \underset{\rho\to\infty}{\to} \chi(\hat{\rho}) + e^{\pm\frac{1}{4}i\pi} E^{\frac{3}{4}} 3^{-\frac{3}{2}} (4\pi)^{-1} \rho^{-\frac{5}{2}} \times \langle \pm E^{\frac{1}{2}} \hat{\rho}_u | T^\pm(E) | \chi\rangle e^{\pm i\rho E^{1/2}}, \quad (8)$$

where we emphasize that all pairs are well separated in the limit. This result is completely analogous to the usual result for two-particle scattering. In (8) we have also given the corresponding asymptotic form of $|\psi^-\rangle$.

If, however, ρ is large but one pair is still close together, we cannot expect that (8) will be correct. This case may be studied by keeping $\vec{Y}_i$ fixed (for some i) and letting $|\vec{X}_i| \to \infty$. Equation (4) is suitable for investigating this limit. We write $\vec{X}_i = X_i \vec{X}_{iu}$ and let $X_i \to \infty$, and, using the same method, find that

$$\psi^\pm(\hat{\rho}) \underset{X_i\to\infty}{\to} \chi(\hat{\rho}) - 3^{-\frac{3}{2}}(4\pi)^{-1}(X_i)^{-1}\{\int d\vec{Q}_i \psi_{\vec{Q}_i}{}^\mp(\vec{Y}_i) \exp[\pm iX_i P(\vec{Q}_i{}^2)]$$
$$\times \langle \pm P(\vec{Q}_i{}^2)\vec{X}_{iu}, \vec{Q}_i | T^\pm(E) | \chi\rangle + 2^{\frac{3}{2}} \sum_{\alpha_i} \chi_\alpha{}^i(\vec{Y}_i) \exp[\pm iX_i P(-E_\alpha{}^i)] \langle \chi_\alpha{}^i, \pm P(-E_\alpha{}^i)\vec{X}_{iu} | T^\pm(E)\chi\rangle\}, \quad (9)$$

where $P(z) = (E-z)^{1/2}$, and we have used the equality of $\langle \hat{K} | T^\pm(E) | \chi\rangle$ and $\langle \hat{K}_i | \overline{T}_i{}^\pm | \chi\rangle$ on the energy shell. The integral over $\vec{Q}_i$ is restricted to those $\vec{Q}_i$ satisfying $\vec{Q}_i{}^2 \leq E$.

The form given in (9) may be further simplified by noting that the phase of $\exp[iX_i P(Q_i{}^2)]$ is stationary at $Q_i = 0$. Near $Q_i = 0$, $\psi_{\vec{Q}_i}{}^-(\vec{Y}_i)$ and the on-shell T-matrix element are analytic, so that the previous techniques give

$$\psi^{\pm}(\hat\rho) \underset{X_i\to\infty}{\to} \chi(\hat\rho)-(\tfrac{2}{3})^{\frac{3}{2}}(4\pi)^{-1}(X_i)^{-1}\{-e^{\pm i\frac{1}{4}\pi}E^{\frac{3}{4}}\pi^{\frac{3}{2}}X_i^{-\frac{3}{2}}\psi_O^{\mp}(\vec{Y}_i)\langle\pm E^{\frac{1}{2}}\vec{X}_{iu},O|T^{\pm}(E)|\chi\rangle\exp(\pm iX_iE^{\frac{1}{2}})$$

$$+\sum_{\alpha_i}\chi_\alpha{}^i(\vec{Y}^i)\exp[\pm iX_iP(-E_\alpha{}^i)]\langle\chi_\alpha{}^i,\pm P(-E_\alpha{}^i)\vec{X}_{iu}|T^{\pm}(E)|\chi\rangle\},\qquad(10)$$

where $\vec{Y}_i$ is fixed as $X_i\to\infty$.

Thus we have seen that the asymptotic form of $\psi^{\pm}(\hat\rho)$ for any $\hat\rho$ with ρ large is determined by the on-shell scattering matrix from the initial state to all possible final states, assuming we regard the two-body wave functions $\psi_Q{}^-$ and $\chi_\alpha{}^i$ as known.

We may now follow the standard technique, described for instance in Ref. 1, to obtain a variational principle of the Kohn[8] type. We study the variation of the integral I defined by

$$I=\int d\hat\rho\,\psi^{-*}(\hat\rho)(E-H)\psi^{+}(\hat\rho),\qquad(11)$$

and ψ^+,ψ^- are varied about their correct values, but maintaining the asymptotic forms (8) and (10) with some unknown on-shell T-matrix elements. We find that

$$\delta I=\int d\hat\rho(\psi^{-*}\nabla_\rho{}^2\delta\psi^{+}-\delta\psi^{+}\nabla_\rho{}^2\psi^{-*})$$

$$=\int_\Sigma d\hat{S}\cdot(\psi^{-*}\hat\nabla\delta\psi^{+}-\delta\psi^{+}\hat\nabla\psi^{-*}),\qquad(12)$$

where Σ is the boundary of the six-dimensional volume of integration, and must be taken to ∞. For Σ, we shall take the "sphere" $\rho^2=R^2$ except near $\vec{Y}_i=0$, where we shall use spherical cylinders $\vec{X}_i{}^2=R'^2$ which intersect the sphere at $\vec{Y}_i{}^2=R^2-R'^2$. We assume that $(R^2-R'^2)$ approaches infinity as $R\to\infty$ in such a way that the solid angle subtended by the nonspherical parts Σ_i of Σ tends to 0.

The only nonzero contribution to δI (in the limit) comes from taking χ of ψ^- with the $\chi_1{}^1$ part of (10). This leads, using $d\hat{S}\cdot\hat\nabla=X_1{}^2d\vec{Y}_1d\vec{X}_1\partial/\partial X_1$, to

$$\delta I=-(\tfrac{2}{3})^{\frac{3}{2}}(4\pi)^{-1}(2\pi)^{-3}\int_{\Sigma_1}d\vec{Y}_1d\vec{X}_{1u}X_1\chi_1{}^{1*}(\vec{Y}_1)\chi_1{}^1(\vec{Y}_1)e^{-i\vec{P}'\cdot\vec{X}_1}e^{iP'X_1}[i(P'+\vec{P}'\cdot\vec{X}_{1u})-X_1{}^{-1}]$$

$$\times\delta\langle\chi_1{}^1,P'\vec{X}_{iu}|T^{+}(E)|\chi\rangle=-3^{-\frac{3}{2}}(2\pi)^{-3}\delta\langle\chi_1{}^1,\vec{P}'|T^{+}(E)|\chi\rangle.\qquad(13)$$

This may be seen by making a partial wave expansion in $\vec{X}_1$. We have used the fact that the volume of integration for $\vec{Y}_1$ approaches infinity, so that we obtain the normalization integral

$$\int d\vec{Y}_1|\chi_1{}^1(\vec{Y}_1)|^2=2^{-3/2}.\qquad(14)$$

The other contributions to δI from Σ_i may be shown to be 0 in a similar manner.

From the spherical part of Σ, leaving aside the contributions of χ to ψ^- (which obviously gives 0), we may use the asymptotic form (8) and $d\hat{S}\cdot\hat\nabla=\rho^5d\hat\rho_u\partial/\partial\rho$ to find

$$\delta I=e^{\frac{1}{2}i\pi}E^{\frac{3}{2}}3^{-3}(4\pi)^{-2}\int_{\Sigma_{\rm sph}}d\hat\rho_u[iE^{\frac{1}{2}}-iE^{\frac{1}{2}}]$$

$$\times e^{2iRE^{1/2}}\langle-E^{\frac{1}{2}}\hat\rho_u|T^{-}(E)|\chi\rangle^*\delta\langle E^{\frac{1}{2}}\hat\rho_u|T^{+}(E)|\chi\rangle=0,\qquad(15)$$

disregarding terms which vanish in the limit $R\to\infty$.

If $|\psi^+\rangle$ and $|\psi^-\rangle$ referred to different initial two-body bound states, say $|\chi_a\rangle$ and $|\chi_b\rangle$, then (13) would read

$$\delta\{\int d\hat{\rho}\,\psi_b^{-*}(\hat{\rho})(E-H)\psi_a^{+}(\hat{\rho}) + 3^{-3/2}(2\pi)^{-3}\langle\chi_b|T^+|\chi_a\rangle\}=0, \quad (16)$$

which has the form of the Kohn[8] variational principle.

Our analysis of the asymptotic form of the wave function was based on the analytic properties of the T matrix near real momenta. The properties would hold for any potential whose matrix elements are analytic near real momenta which behaved suitably for large momenta. It is not necessary to insist on a superposition of Yukawa potentials.

We believe that this variational principle may provide a practical, efficient method of calculating scattering amplitudes for $(2\to 3)$ processes. It is superior to techniques involving integral equations in that $(3\to 3)$ processes along with their awkward singularities are not explicitly involved. It remains to apply the method to an actual problem.

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SPIN AND PARITY OF THE $N_{3/2}^*(2420)$

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The preliminary results of an elastic-scattering experiment of positive π mesons on protons have recently been obtained. The work was performed at the Rutherford High Energy Laboratory. The differential cross sections have been measured at ten incident pion momenta between 1.72 and 2.8 GeV/c and a qualitative analysis is presented which suggests that the enhancement seen in the π^+p total cross section at a center-of-mass energy of 2420 MeV[1] may be explained by an $11/2^+$ resonance.

The experimental details are similar to those of earlier work using negative π mesons[2] and will be published with the final differential cross sections in the near future.

For the present analysis the differential cross sections were normalized in the following way. An exponential function $y=a_0e^{a_1t}$ was fitted over the region of the square of the momentum transfer $-t<0.4$ (GeV/c)^2. This was extrapolated to $t=0$ and matched to the forward cross section calculated from the optical theorem and dispersion relations.[3] This procedure gives a relative normalization error of about 1 mb in the total elastic cross section. Absolute normalization of the results is being undertaken at present.

The differential cross sections have been fitted by a series of Legendre polynomials

$$d\sigma/d\Omega = \lambda\!\!\!^-{}^2 \sum_0^{N_{\max}} C_n P_n(\cos\theta)$$

by the method of least squares. The coefficients C_n which were obtained are shown as functions of incident pion momentum in Fig. 1. The high-