ASYMPTOTIC SU(3) AND VECTOR MESON DECAYS*

T. Das,† V. S. Mathur, and S. Okubo
Department of Physics and Astronomy, University of Rochester, Rochester, New York
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We have derived several sum rules based upon the assumption that the SU(3) group will be exact at infinite energy. In this way, decays of vector mesons into lepton pairs have been computed together with the decay rate for $\varphi \to K\bar{K}$.

Recently, it has been shown[1,2] that the study of the asymptotic behavior of suitable linear combinations of matrix elements constructed on the basis of some symmetry can lead to useful results regarding broken symmetry. In the present note, we show how the same idea can be applied to obtain some interesting results for vector meson decays.

We define the following propagator functions:

$$\Delta_{\mu\nu}^{(\alpha)}(k) = i\int d^4x\, e^{ik\cdot x}\langle 0|T\{V_\mu^{(\alpha)}(x)V_\nu^{(\alpha)}(0)\}|0\rangle, \quad (1)$$

where

$$V_\mu^{(\alpha)}(x) = \tfrac{1}{2}i\bar{q}(x)\gamma_\mu\lambda_\alpha q(x) \quad (\alpha = 0, 1, \cdots, 8) \quad (2)$$

in terms of the quark field $q(x)$. Assuming that the SU(3) symmetry becomes exact at $k \to \infty$, i.e.,

$$\lim_{k\to\infty} [\Delta_{\mu\nu}^{(3)}(k) - \Delta_{\mu\nu}^{(8)}(k)] = 0, \quad (3)$$

and following the same procedure as in Ref. 2, we immediately obtain the sum rule

$$\int dm^2 \frac{\rho_8(m^2)}{m^2} = \int dm^2 \frac{\rho_3(m^2)}{m^2}, \quad (4)$$

where

$$\Delta_{\mu\nu}^{(\alpha)}(k) = \int dm^2 \left[\delta_{\mu\nu}\rho_\alpha(m^2) + \frac{k_\mu k_\nu}{m^2}\rho_\alpha(m^2)\right]\frac{1}{k^2+m^2} + \text{Schwinger terms}. \quad (5)$$

If we now assume that the spectral functions are dominated by ρ^0, ω, and φ mesons, we get

$$G_\rho^2/m_\rho^2 = G_\omega^2/m_\omega^2 + G_\varphi^2/m_\varphi^2, \quad (6)$$

where G_ρ is defined by

$$\langle 0|V_\mu^{(3)}(0)|\rho^0(q)\rangle = \epsilon_\mu(q)G_\rho/(2q_0V)^{1/2} \quad (7)$$

and G_ω (G_φ) is defined similarly, replacing $V_\mu^{(3)}$ and ρ by $V_\mu^{(8)}$ and ω (φ), respectively. Equation (6) gives a sum rule among the partial rates[3] of the vector mesons decaying into lepton pairs, namely

$$\tfrac{1}{3}m_\rho\Gamma(\rho^0 \to l\bar{l}) = m_\omega\Gamma(\omega \to l\bar{l}) + m_\varphi\Gamma(\varphi \to l\bar{l}), \quad (8)$$

where l corresponds to μ or e and we have assumed as usual that the electromagnetic current j_μ is given by $j_\mu = V_\mu^{(3)} + V_\mu^{(8)}/\sqrt{3}$.

Using the experimental branching ratios of $\rho^0 \to \mu^+\mu^-$[4] and $\omega \to e^+e^-$[5] which are $(5.1\pm1.2)\times10^{-5}$ and $(12\pm3)\times10^{-5}$, respectively, we find[5] $\Gamma(\varphi\to\mu^+\mu^-)/\Gamma(\varphi\to\text{all}) \simeq 1.6\times10^{-4}$ which is consistent with the experimental upper limit[6] $\simeq 7.4 \times 10^{-4}$. For a better check of sum rule (8) we have to await future experiments.

If we further assume a supervalidity of the SU(3) for $k\to\infty$, i.e., if ρ_α satisfies also the superconvergent sum rule

$$\int dm^2[\rho_3(m^2) - \rho_8(m^2)] = 0, \quad (9)$$

then we would get a superconvergent sum rule retaining as before the ρ, ω, and φ contributions:

$$G_\rho^2 = G_\omega^2 + G_\varphi^2. \quad (10)$$

It has been emphasized in Ref. 2 that the information content on symmetry breaking becomes less and less with the assumption of stronger and stronger conditions of superconvergence. Thus one expects relation (8) or (6) to be satisfied much better than the relation (10). In fact for the experimental masses of ρ, ω, and φ, Eq. (10) is inconsistent[7] with Eq. (6). To remedy this situation and to obtain more information on symmetry breaking than given by Eq. (6), we adopt the point of view that we have to take into account the SU(3)-violating effects in order to obtain higher order superconvergent sum rules. For instance, we should consider the first-order SU(3) symmetry-breaking interaction ($T^{(8)}$) in order to derive the first-order superconvergent sum rule instead of Eq. (9). For the second-order superconvergent sum rule we need to take account of the second-order effect of symmetry breaking, and so on. If this idea is correct, then we should replace Eq. (9) now by

$$\int dm^2\{\rho_3(m^2)+3\rho_8(m^2)-4\rho_4(m^2)\}=0, \qquad (11)$$

since the left-hand side of Eq. (11) is zero to the first order in $T^{(8)}$. Now, Eq. (11) leads to

$$G_\omega{}^2+G_\varphi{}^2=\tfrac{1}{3}(4G_{K^*}{}^2-G_\rho{}^2). \qquad (12)$$

From the asymptotic condition

$$\lim_{k\to\infty}[\Delta_{\mu\nu}{}^{(3)}(k)-\Delta_{\mu\nu}{}^{(4)}(k)]=0, \qquad (13)$$

we also obtain the relation

$$G_{K^*}{}^2/m_{K^*}{}^2=G_\rho{}^2/m_\rho{}^2, \qquad (14)$$

where we have retained the ρ and K^* contributions only and have neglected the contribution of the κ meson. Experimentally the existence of κ is doubtful. However, even if it exists, we know that G_κ is of first order in SU(3) breaking, so that the contribution of the κ meson to Eq. (14) which enters as a term proportional to $G_\kappa{}^2$ will be at least of second order in SU(3) breaking, and may be neglected.

Now we can determine G_ω, G_φ, and G_{K^*} in terms of G_ρ by means of Eqs. (6), (12), and (14):

$$G_\varphi{}^2=\frac{m_\varphi{}^2(4m_{K^*}{}^2-m_\rho{}^2-3m_\omega{}^2)}{3m_\rho{}^2(m_\varphi{}^2-m_\omega{}^2)}G_\rho{}^2\simeq1.03G_\rho{}^2,$$

$$G_\omega{}^2=\frac{m_\omega{}^2(3m_\varphi{}^2+m_\rho{}^2-4m_{K^*}{}^2)}{3m_\rho{}^2(m_\varphi{}^2-m_\omega{}^2)}G_\rho{}^2\simeq0.43G_\rho{}^2,$$

$$G_{K^*}{}^2=(m_{K^*}{}^2/m_\rho{}^2)G_\rho{}^2 \simeq1.34G_\rho{}^2. \qquad (15)$$

We then compute

$$\frac{\Gamma(\omega\to l\bar{l})}{\Gamma(\rho^0\to l\bar{l})}\simeq0.14,\quad \frac{\Gamma(\varphi\to l\bar{l})}{\Gamma(\rho^0\to l\bar{l})}\simeq0.15. \qquad (16)$$

Using the experimental branching ratio[4] $\Gamma(\rho\to\mu^+\mu^-)/\Gamma(\rho^0\to\text{all})=(5.1\pm1.2)\times10^{-5}$, we obtain from Eq. (16) the branching ratios for ω and φ leptonic decays as

$$\frac{\Gamma(\omega\to l\bar{l})}{\Gamma(\omega\to\text{all})}=(8.4\pm3.0)\times10^{-5},$$

$$\frac{\Gamma(\varphi\to l\bar{l})}{\Gamma(\varphi\to\text{all})}=(2.7\pm1.3)\times10^{-4}. \qquad (17)$$

These results should be compared with the experimental branching ratio of $(12\pm3)\times10^{-5}$ for $\omega\to e^+e^-$ [5] and the upper limit of 7.4×10^{-4} for $\varphi\to\mu\bar{\mu}$.[6] Finally, the branching ratio for the leptonic decay mode of the ρ^0 can also be compared if we use the current-algebra result[8]

$$G_\rho{}^2/m_\rho{}^2=f_\pi{}^2=G_\rho/g_{\rho\pi\pi}, \qquad (18)$$

where f_π is the π-decay constant. One then obtains

$$\frac{\Gamma(\rho^0\to l\bar{l})}{\Gamma(\rho^0\to\text{all})}=4.0\times10^{-5}, \qquad (19)$$

which is also in reasonable agreement with the experimental value.

We now turn our attention to the strong decays of vector mesons. The K^* width has been calculated in Ref. 2, in good agreement with the experimental result. Here we confine our attention to the calculation of the decay mode $\varphi\to K\bar{K}$. According to our philosophy we should have

$$\lim_{k\to\infty}\int d^4x\, e^{ik\cdot x}\langle0|T\{V_\mu{}^{(8)}(x)V_\nu{}^{(0)}(0)\}|0\rangle=0, \qquad (20)$$

since in the exact SU(3) limit this is an identity. Here $V_\mu{}^{(0)}$ corresponds to the unitary-singlet vector current with $\lambda_0=\sqrt{\tfrac{2}{3}}$. Defining σ_ω by

$$\langle0|V_\mu{}^{(0)}(0)|\omega(q)\rangle=\sigma_\omega\epsilon_\mu(q)/(2q_0V)^{1/2}, \qquad (21)$$

and similarly σ_φ by replacing ω by φ, one derives the following sum rule in the same approximation of saturating the intermediate states by ω and φ poles:

$$\sigma_\omega G_\omega/m_\omega{}^2+\sigma_\varphi G_\varphi/m_\varphi{}^2=0. \qquad (22)$$

On the other hand, in the exact SU(3) limit, we must have $\langle K(p')|V_\mu{}^{(0)}|K(p)\rangle=0$. Thus, when one sets

$$\langle K(p')|V_\mu{}^{(0)}(0)|K(p)\rangle =(4p_0p_0{}'V^2)^{-1/2}(p+p')_\mu F(k^2), \qquad (23)$$

with $k^2=(p-p')^2$, one expects

$$\lim_{k\to\infty} F(k^2)=0,$$

so that $F(k^2)$ should satisfy an unsubtracted dispersion relation. Saturating intermediate states again by ω and φ poles and noting that $F(0)=0$, since the operator $\int d^3x\, V_4^{(0)}(x)$ is proportional to the baryon number which must be exactly conserved, one finds now

$$0=\sigma_\omega g_{\omega K\bar{K}}/m_\omega^{\;2}+\sigma_\varphi g_{\varphi K\bar{K}}/m_\varphi^{\;2}, \tag{24}$$

where $g_{\omega K\bar{K}}$ ($g_{\varphi K\bar{K}}$) is the coupling constant for ω (φ) to $K\bar{K}$. From Eqs. (22) and (24) one obtains

$$g_{\omega K\bar{K}}/g_{\varphi K\bar{K}}=G_\omega/G_\varphi. \tag{25}$$

Moreover, if the usual electromagnetic form factor of the kaon is dominated by vector meson poles then we must have in addition

$$G_\omega g_{\omega K\bar{K}}/m_\omega^{\;2}+G_\varphi g_{\varphi K\bar{K}}/m_\varphi^{\;2}=\tfrac{1}{2}\sqrt{3}. \tag{26}$$

Together with Eqs. (25), (6), and (18) this leads to

$$g_{\varphi K\bar{K}}=\tfrac{1}{2}\sqrt{3}\left(\frac{m_\rho}{G_\rho}\right)^2 G_\varphi=\tfrac{1}{2}\sqrt{3}G_\varphi/f_\pi^{\;2}. \tag{27}$$

If we set $f_\pi=f_k$, Eq. (27) is nothing but the relation[9] obtained on the basis of the algebra of currents. However, in that derivation we must make a rather unphysical limit $M_K\to 0$ while our new derivation is free from this assumption. We remark also that Eqs. (18) and (27) give us

$$g_{\varphi K\bar{K}}/g_{\rho\pi\pi}=\tfrac{1}{2}\sqrt{3}G_\varphi/G_\rho, \tag{28}$$

which reduces to the standard SU(3) relation if we replace G_φ/G_ρ by $\cos\theta$, where θ is the mixing angle. From Eq. (28) and using the result (15), we compute

$$\Gamma(\varphi\to K\bar{K})\simeq 5.0 \text{ MeV}, \tag{29}$$

which must be compared with the experimental value of (4 ± 1) MeV.

In our approach, we never make use of the so-called ω-φ mixing theory.[10] However, it may be worthwhile to point out that Eq. (6) contains information on this aspect. Indeed, in the course of deriving Eq. (6) we might as well have saturated the intermediate states by ω_8 and ω_1, rather than φ and ω, where ω_8 and ω_1 correspond to pure unitary-octet and -singlet vector mesons, respectively. Then we would get the relation

$$G_\omega^{\;2}/m_\omega^{\;2}+G_\varphi^{\;2}/m_\varphi^{\;2}=G_8^{\;2}/m_8^{\;2}+G_1^{\;2}/m_1^{\;2}, \tag{30}$$

where G_8 and G_1 are defined analogously as in Eq. (7). Using the notion of the ω-φ mixing theory, one can rewrite this equation as

$$\left(\frac{\sin^2\theta}{m_\omega^{\;2}}+\frac{\cos^2\theta}{m_\varphi^{\;2}}-\frac{1}{m_8^{\;2}}\right)G_8^{\;2}+\left(\frac{\cos^2\theta}{m_\omega^{\;2}}+\frac{\sin^2\theta}{m_\varphi^{\;2}}-\frac{1}{m_1^{\;2}}\right)G_1^{\;2}=\sin 2\theta\,\frac{m_\varphi^{\;2}-m_\omega^{\;2}}{m_\varphi^{\;2}m_\omega^{\;2}}G_8G_1. \tag{31}$$

Now, if λ is the measure of the SU(3) violation, then $G_1=O(\lambda)$ and hence $G_1^{\;2}$ can be neglected up to the first order in λ. Also, $(m_\varphi^{\;2}-m_\omega^{\;2})$ may be regarded as a measure of a deviation from the exact nonet symmetry[11] [or W(3) or SU(6) symmetry] and we shall denote it to be of the order λ'. Therefore, if one can neglect terms of the order λ^2 and $\lambda\lambda'$, then the above equation reduces to

$$\frac{\sin^2\theta}{m_\omega^{\;2}}+\frac{\cos^2\theta}{m_\varphi^{\;2}}=\frac{1}{m_8^{\;2}}+O(\lambda^2,\lambda\lambda'). \tag{32}$$

This is identical to the formula derived by Coleman and Schnitzer[12] some years ago, which gives $\theta\simeq 34°$. Note that in this equation, the mass terms appear in the denominator rather than in the numerator in contrast to the usual method.[10,11].

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†On leave from Tata Institute of Fundamental Research, Bombay, India.

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[7]Of course the symmetry result satisfies both Eqs. (6) and (10), as it must. It is also interesting to note that if we take the experimental near equality of the ρ and ω masses as an exact degeneracy, we have a possible solution of Eqs. (6) and (10) with $G_\varphi = 0$. This is the point of view taken by S. L. Glashow, H. J. Schnitzer, and S. Weinberg, Phys. Rev. Letters 19, 139 (1967). Our philosophy is entirely different from theirs since we reject Eq. (9) or (10) in favor of Eq. (11). Equation (6) has also been derived recently by P. P. Divakaran and L. K. Pandit (to be published), as well as by G. C. Joshi and L. K. Pande (to be published).

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KOHN VARIATIONAL PRINCIPLE FOR THREE-PARTICLE SCATTERING*

J. Nuttall
Texas A & M University, College Station, Texas
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The Kohn variational principle is extended to apply to scattering processes where a two-particle bound state is broken up by a third particle.

In the past it has often proved that use of a variational principle[1] has been the most efficient method of making quantitative calculations of the properties of systems of several particles (more than two). In particular, the Kohn-Hulthén principle has been applied successfully to problems where the open channels contain only two bound parts, as for example, electron-hydrogen atom scattering.[2] In this Letter, we discuss how the principle may be extended to include open channels containing three separated particles.

As an example, we take the case of three different particles of equal mass ($m = \frac{1}{2}$) interacting through two-body potentials that are superpositions of Yukawa potentials. We assume that each pair may form a number of bound states labeled by α_i with binding energies $E_\alpha{}^i$ ($i = 1, 2, 3$ denotes the particle that is not bound). In practice the initial state of the system will contain one of these bound pairs, say 2 and 3, and will be described by the state $|\chi\rangle = |\chi_1{}^1, \vec{p}'\rangle$, an eigenstate of $H_1 = H_0 + V_1$, energy E, where H_0 is the kinetic energy of all three particles and V_1 the interaction between 2 and 3. We work in the frame with total momentum zero, and $\vec{p}'$ is the initial momentum of particle 1.

Using Eq. (184), p. 102, of Goldberger and Watson[3] we may write the scattering wave function corresponding to initial state $|\chi\rangle$ as

$$\psi^+(\hat{\rho}_i) = \chi(\hat{\rho}_i) + (2\pi)^{-9/2} 3^{-3/2} \int d\hat{K}_i \exp(i\hat{\rho}_i \cdot \hat{K}_i)(E - \hat{K}^2 + i\epsilon)^{-1} \langle \hat{K}_i | T^+(E) | \chi \rangle, \tag{1}$$

where we have used a notation similar to Lovelace[4] such that $\hat{\rho}_i$ is a six-dimensional vector

$$\hat{\rho}_1 = (\vec{X}_1, \vec{Y}_1) = \{(\tfrac{2}{3})^{1/2}[\vec{r}_1 - \tfrac{1}{2}(\vec{r}_2 + \vec{r}_3)], (2)^{-1/2}[\vec{r}_2 - \vec{r}_3]\} \tag{2}$$

and

$$\hat{K}_1 = (\vec{P}_1, \vec{Q}_1) = \{(\tfrac{2}{3})^{1/2}\vec{p}_1, (2)^{-1/2}[\vec{p}_2 - \vec{p}_3]\} \tag{3}$$

and similarly for $i = 2, 3$.

Alternatively, proceeding from Eq. (81), p. 79 of Ref. 3, taking the $\chi_b{}^i$ as eigenstates of $H_i = H_0 + V_i$, we find

$$\psi^+(\hat{\rho}_i) = \chi(\hat{\rho}_i) + (2\pi)^{-3} 3^{-\frac{3}{2}} \int d\hat{K}_i \psi_{\vec{Q}_i}{}^-(\vec{Y}_i) \exp(i\vec{X}_i \cdot \vec{P}_i)(E - \hat{K}^2 + i\epsilon)^{-1} \langle \hat{K}_i | \overline{T}_i{}^+ | \chi \rangle$$
$$+ (2\pi)^{-3} (\tfrac{2}{3})^{\frac{3}{2}} \sum_{\alpha_i} \int d\vec{P}_i \chi_\alpha{}^i(\vec{Y}_i) \exp(i\vec{X}_i \cdot \vec{P}_i)(E + E_\alpha{}^i - \vec{p}_i{}^2 + i\epsilon)^{-1} \langle \chi_\alpha{}^i, \vec{P}_i | T(E) | \chi \rangle. \tag{4}$$