Rosenfeld et al., Rev. Mod. Phys. 39, 1 (1967)] with a cutoff at $\overline{\nu_L}=2.4$ BeV and at 1.13 BeV. This reproduces the results (1)-(6) qualitatively. Note that a high cutoff is bad in the resonance model. The established resonances become a very poor approximation to the full amplitude, because their elasticities decrease exponentially. Compare K. Igi and S. Matsuda, Phys. Rev. (to be published), who were the first to apply S_1 to $B^{(-)}$ at $t=0$ using the resonance model. They obtained the ratio $\nu B^{(-)}/A^{(-)}$ too small by a factor of 2 because of their high cutoff at $\nu_L=5.5$ BeV.

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[19]F. Arbab and C. B. Chiu, Phys. Rev. 147, 1045 (1966).

[20]The error bars of the points II in Fig. 2 include (a) the experimental error in the low-energy integral (Born term and phase shifts) and (b) an estimate of the background integral in the j plane, as derived from the size of the wiggles. This latter error would rapidly diminish with a higher limit of integration N.

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NEW APPROACH TO ALGEBRA OF CURRENTS AND APPLICATION TO $K\to 2\pi$ DECAY*

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Ambiguities arising in some applications of current algebra are overcome by employing the algebra of currents to calculate the subtraction constant in a once-subtracted dispersion relation. The new approach is applied to $K\to 2\pi$ decay and yields corrections to the usual current-algebra method of about 10%. The branching ratio $(K^+\to\pi^++\pi^0)/(K_1^0\to\pi^++\pi^-)$ is also computed and agrees well with experiment.

It has been demonstrated by several authors[1] that the $\Delta I=\frac{1}{2}$ rule governing nonleptonic decays of kaons follows naturally from the algebra of currents, even though the original "current-current" Hamiltonian may contain an intrinsic $\Delta I=\frac{3}{2}$ part. However, one should note that the application of the soft-pion technique to $K\to 2\pi$ decay implies setting $m_K=m_\pi$ because of energy-momentum conservation. Hence, the $\Delta I=\frac{1}{2}$ rule for $K\to 2\pi$ decay is only valid in the approximation of neglecting the K-π mass difference. Thus one may inquire whether a large $\Delta I=\frac{3}{2}$ contribution will result if terms of order $[m(K)-m(\pi)]$ are not neglected. On the other hand, we should like the admixture of $\Delta I=\frac{3}{2}$ to be sufficient to explain the mode $K^+\to\pi^++\pi^0$. In this connection, we recall that Nambu and Hara,[2] using the soft-pion method, derived the relation

$$R=\frac{M(K^+\to\pi^++\pi^0)}{M(K_1^0\to\pi^++\pi^-)}\simeq\frac{m^2(\pi^+)-m^2(\pi^0)}{2m^2(K)}\approx\frac{1}{370},\qquad(1)$$

which is too small (the experimental value is $R\simeq 1/22$). The theoretical prediction would be much improved if one could justify the replacement of $m(K)$ by $m(\pi)$ in Eq. (1).

In this note, we take a closer look at the class of decays wherein taking the soft-pion limit imposes an unreasonable constraint on the four-momentum of the decaying particle. To this end, we propose a new way of utilizing the algebra of currents in combination with a once-subtracted dispersion relation. In this new approach we do not take the limit $k\to 0$ (k is the pion four-momentum) but instead let $k^2\to 0$ ($k\to 0$ implies $k^2\to 0$ but not the converse). With this modification of the soft-pion technique, one is able to evaluate the corrections to the $K_1^0\to 2\pi$ calculation of Suzuki and Sugawara[1] and, moreover, one finds that Eq. (1) is replaced by

$$|R|=\left|\frac{M(K^+\to\pi^++\pi^0)}{M(K_1^0\to\pi^++\pi^-)}\right|=\left|\frac{m^2(\pi^+)-m^2(\pi^0)}{2m^2(\pi)}\right|\approx\frac{1}{30}.\qquad(2)$$

We proceed to explain the method; let us set

$$M(K(p)\to\pi_\alpha(k)+\pi_\beta(k'))=\frac{i}{V^{3/2}}\frac{1}{(8p_0k_0k_0')^{1/2}}T_{\alpha\beta}(k^2,k'^2,p^2).\qquad(3)$$

Note that due to energy-momentum conservation $p=k+k'$; $T_{\alpha\beta}$ is a function of the variables p^2, k^2, and k'^2. Hereafter, we always take the mass value $p^2=m^2(K)$, and hence we shall no longer mention the possible dependence on p^2. Our procedure depends upon taking the successive limits $k^2 \to 0$ and $k'^2 \to 0$ separately.

Next, note that

$$T_{\alpha\beta}(k^2, k'^2) = i(4k_0 p_0 V^2)^{\frac{1}{2}} \frac{k'^2 + m^2(\pi)}{f m^2(\pi)} \int d^4x\, e^{-ik'x} \langle \pi_\alpha(k) | \theta(x_0) [\partial_\mu a_\mu^{(\beta)}(x), H_w(0)] | K(p) \rangle, \tag{4}$$

where we have used the condition of partially conserved axial-vector current,

$$i\partial_\mu a_\mu^{(\beta)}(x) = m_\pi^{\ 2} f \pi_\beta(x). \tag{5}$$

In order to evaluate Eq. (4), we make the following crucial observation. If we give up energy-momentum conservation temporarily, the right-hand side of Eq. (4) defines a new function $F_{\alpha\beta}(\nu, t, \Delta, k^2, k'^2)$, where ν, t, and Δ denote

$$\nu = k' \cdot \tfrac{1}{2}(p+k), \quad t = k' \cdot (p-k), \quad \Delta = (p-k)^2. \tag{6}$$

For the actual decay problem where energy and momentum are conserved, these variables take the values $\nu = -\frac{1}{2}[k^2 + m^2(K)]$, $t = \Delta = k'^2$, so that

$$T_{\alpha\beta}(k^2, k'^2) = F_{\alpha\beta}(\nu = -\tfrac{1}{2}[k^2 + m^2(K)], t = k'^2, \Delta = k'^2, k^2, k'^2). \tag{7}$$

For the sake of definiteness, we first calculate Eq. (4) for the case $k'^2 = 0$, $k^2 = -m^2(\pi)$. We may hereafter drop the dependence of $F_{\alpha\beta}$ on k^2 and k'^2. (Note that $F_{\alpha\beta}$ represents the amplitude for the reaction $S + K \to \pi_\alpha + \pi_\beta$, where S is a fictitious spurion; here ν and Δ may be interpreted to be the energy variable and the square of the momentum transfer of the spurion. The "mass" of the spurion is also a variable and, in fact, t may be expressed as a linear combination of Δ and the spurion mass.)

We now assume that $F_{\alpha\beta}$ satisfies a once-subtracted dispersion relation in the energy variable ν with the other variables fixed at $t = 0$, $\Delta = 0$. Thus we have

$$F_{\alpha\beta}(\nu, t=0, \Delta=0) = F_{\alpha\beta}(\nu=0, t=0, \Delta=0) + \frac{\nu}{\pi}\int_{-\infty}^{\infty} d\nu' \frac{1}{\nu' - \nu - i\epsilon} \frac{1}{\nu'} \operatorname{Im} F_{\alpha\beta}(\nu', t=0, \Delta=0). \tag{8}$$

The important point to observe is that the subtraction term $F_{\alpha\beta}(\nu=0, t=0, \Delta=0)$ can be computed from Eq. (4) by formally setting $k'=0$ since $k'=0$ implies $\nu = t = k'^2 = 0$. Integrating by parts with respect to x, one finds

$$F_{\alpha\beta}(\nu=0, t=0, \Delta=0) = -\frac{(4k_0 p_0 V^2)^{1/2}}{f} \langle \pi_\alpha(k) | [V_\beta, H_w(0)] | K(p) \rangle, \tag{9}$$

where the right-hand side of Eq. (9) must be evaluated at the point $\Delta = (p-k)^2 = 0$. In the derivation of Eq. (9), we assumed[3] the usual equal-time commutation relation so that V_β stands for the generator of the SU(2) group.

One immediately recognizes that the first term on the right-hand side of Eq. (8) is exactly what one expects from the usual application of the algebra of currents,[4] while the remaining dispersion integral represents the correction proportional to $\nu = \frac{1}{2}[m^2(\pi) - m^2(K)]$. One can estimate this correction term by computing the contributions from the K^* and ρ intermediate states in the dispersion integral.[5] For this purpose, one needs to know the matrix elements

$$\langle \pi_\alpha(k) | H_w(0) | K^*(q) \rangle, \quad \langle \rho(q) | H_w(0) | K(p) \rangle. \tag{10}$$

It is evident that the $\Delta I = \frac{3}{2}$ part of $H_w(0)$ can now contribute. However, for the first matrix element, one may still take the soft-pion limit $k^2 \to 0$. This is done by essentially repeating the earlier argument with a slight modification and it turns out that the equal-time commutator gives zero contribution. Further, the dispersion contribution to this matrix element may be estimated by saturating it by the K pole. One finds in this way that the K^*-pole contribution to the second term on the right-hand side of Eq. (8) yields a purely $\Delta I = \frac{1}{2}$ correction of the order of $[m^2(K) - m^2(\pi)]/2m^2(K^*) \approx 10\%$ compared with the first term $F_{\alpha\beta}(\nu = 0, t = 0, \Delta = 0)$. Unfortunately, we cannot use the same method for $\langle \rho(q) | H_w(0) | K(p) \rangle$ unless we let $m(K) \to 0$. Thus, in principle, for this matrix element, the $\Delta I = \frac{3}{2}$ part may be non-negligible. However, the ρ meson pole does not give any contribution if we symmetrize our matrix element with respect to the two final pions and if we neglect the π^+-π^0 mass difference. Hence, we do not expect the contribution of the dispersion integral to $\Delta I = \frac{3}{2}$ to exceed its contribution to $\Delta I = \frac{1}{2}$. Thus, with a maximum 20% error, we may neglect the dispersion contribution in Eq. (8) and write

$$\lim_{k'^2 \to 0} T_{\alpha\beta}(k^2, k'^2) \approx F_{\alpha\beta}(\nu = 0, t = 0, \Delta = 0). \qquad (11)$$

Note that whereas Eq. (11) is exact in the usual limit $k' \to 0$, we have shown that it is also approximately correct in the weaker limit $k'^2 \to 0$.

We go one step further and take the limit $k^2 \to 0$ of $T_{\alpha\beta}(k^2, k'^2)$. One can again use the technique outlined above to evaluate $F_{\alpha\beta}(\nu = 0, t = 0, \Delta = 0)$ given by Eq. (9). First, observe that $F_{\alpha\beta}(\nu = 0, t = 0, \Delta)$ is a function of $s \equiv k \cdot p = \frac{1}{2}[k^2 + p^2 - \Delta]$ and assume a once-subtracted dispersion relation with respect to s. Again, the subtracted term at $s = 0$ can be computed by means of the algebra of currents, while the dispersion integral is now expected to be small since the K^*, ρ, K, and π intermediate states can easily be shown to give zero contribution. Hence, one can set

$$F_{\alpha\beta}(\nu = 0, t = 0, \Delta = 0) \approx -f^{-2}(2p_0)^{1/2} \langle 0 | [V_\alpha, [V_\beta, H_w(0)]] | K(p) \rangle, \qquad (12)$$

which gives a purely $\Delta I = \frac{1}{2}$ transition. Hence, any intrinsic $\Delta I = \frac{3}{2}$ part in the weak Hamiltonian can contribute to $K \to 2\pi$ only through the dispersion integral in Eq. (8), which we have seen is small compared with the subtraction constant or the equal-time commutator term in Eq. (8). Since, from Eq. (12), the latter term leads to a dominantly $\Delta I = \frac{1}{2}$ contribution to $K \to 2\pi$ decay, we conclude that the $\Delta I = \frac{3}{2}$ part of the weak Hamiltonian is indeed suppressed. Clearly, however, in problems where one may be looking at only $\Delta I = \frac{3}{2}$ or higher effects, as for instance in $K^+ \to \pi^+ + \pi^0$ decay, the dispersion integral in Eq. (8) would acquire importance and affords, in principle, a way of performing the calculation.

For $K^+ \to \pi^+ + \pi^0$ decay, we may alternatively assume[6] the possibility that there is no intrinsic $\Delta I = \frac{3}{2}$ part in the weak Hamiltonian and that the decay arises chiefly from the small mass difference between π^+ and π^0. Then, as an approximation, we shall neglect all dispersion integral corrections. Actually, we find that the final answer is not seriously affected by their inclusion. Using Eq. (12) for the final matrix element, we find

$$M(K^+ \to \pi^+ + \tilde{\pi}^0) : M(K^+ \to \pi^0 + \tilde{\pi}^+) : M(K_1^0 \to \pi^+ + \tilde{\pi}^-) : M(K_1^0 \to \pi^+ + \tilde{\pi}^+) : M(K_1^0 \to \pi^0 + \tilde{\pi}^0) = -1:1:1:1:1, \qquad (13)$$

where the symbol $M(K \to \pi^\alpha + \tilde{\pi}^\beta)$ implies that we have set $k_\beta^2 \to 0$. If the physical $M(K \to \pi^\alpha + \pi^\beta)$ is expanded as a linear combination of k_α^2 and k_β^2, the result is

$$R = \frac{M(K^+ \to \pi^+ + \pi^0)}{M(K_1^0 \to \pi^+ + \pi^-)} = \frac{m^2(\pi^0) - m^2(\pi^+)}{2m^2(\pi)},$$

which is Eq. (2). The argument leading to this result is very similar to that of Ref. 2, but we end up with $m^2(\pi)$ instead of $m^2(K)$ in the denominator.[7] It is interesting to observe that several models recently considered lead to the same conclusion.[8] We would like to emphasize that in the usual treatment of Hara and Nambu, Eq. (13) is derived under the more restrictive limit when the four-momentum of the relevant pion itself vanishes. This is the basic difference in our result which leads to the substantial change in the result for the ratio R.

We finally remark that if $F_{\alpha\beta}(\nu, t, \Delta)$ satisfies an unsubtracted dispersion relation and if the dominant contribution comes from the K^* and ρ intermediate states, then the results of the algebra of currents are essentially equivalent to those obtained with the vector-dominance model (as has been conjectured by Sakurai).[9] This is connected also with the validity of a sum rule analogous to that of Fubini–

Dashen–Gell-Mann.[10] Indeed, if $F_{\alpha\beta}(\nu,t,\Delta)$ satisfies an unsubtracted dispersion relation, then

$$\lim_{\nu=\infty} F_{\alpha\beta}(\nu,t,\Delta)=0$$

or, equivalently, we have

$$F_{\alpha\beta}(\nu=0,t=0,\Delta) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\nu' \frac{1}{\nu'}\mathrm{Im}F_{\alpha\beta}(\nu',t=0,\Delta) \quad (14)$$

because of Eq. (8). It is easy to see that Eq. (14) is the analog of the Fubini–Dashen–Gell-Mann sum rule if we recall Eq. (9). It follows that if $F_{\alpha\beta}(\nu,t,\Delta)$ satisfies a once-subtracted dispersion relation [cf. Eq. (8)], the dispersion integral will provide a measure of the deviation from the vector dominance model.

Applications of these techniques to other particle decays are in progress and will be reported elsewhere.

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[1]M. Suzuki, Phys. Rev. Letters 15, 986 (1965); H. Sugawara, Phys. Rev. Letters 15, 870, 997 (1965).

[2]Y. Hara and Y. Nambu, Phys. Rev. Letters 16, 875 (1966).

[3]We can show that the so-called Schwinger term does not contribute if it is a c number.

[4]Equations (8) and (9) may be derived also from the ordinary method of setting $k'\to 0$ if we simultaneously let $p\to\infty$ (and hence $k\to\infty$ due to $p=k+k'$) while keeping $k'\cdot p$, $k\cdot p$, p^2, and k^2 finite subject to $k'^2=(p-k)^2\to 0$.

[5]In our procedure, we contract and extrapolate (to $k^2=0$) the pions one by one in contrast to the technique of doing this "simultaneously." It is well known that in the latter case [S. Weinberg, Phys. Rev. Letters 16, 879 (1966)] one gets extra terms, the so-called σ terms. It has been argued by L. S. Kisslinger [Phys. Rev. Letters 18, 861 (1966)] that for extrapolation to zero pion four-momenta, the second procedure seems preferable since the σ terms contain information on the final-state interaction. However, with a different extrapolation technique ($k^2\to 0$) that we employ, even following the first procedure, the π-π interaction effects are contained in the dispersion integral [see Eq. (8)]. In practice one could use for instance the σ meson to estimate this interaction. However, the existence of σ is uncertain and we feel that the point is not essential to our argument.

[6]Or, we may assume that the intrinsic $\Delta I=\frac{3}{2}$ part effectively gives a contribution of the less than 1%. Then, its presence will be negligible, compared with the contribution from the $\pi^+-\pi^0$ mass differences [see Eq. (2)]. As we have seen, the dispersion integral seems to give corrections of about 10%, most of which may be purely $\Delta I=\frac{1}{2}$. Hence, this possibility need not be discarded.

[7]In order to clarify how our results lead to an enhancement of the ratio R explicitly, we may recall the simple argument used by Hara and Nambu, Ref. 2, where one starts by describing the physical matrix element as a quadratic function of the meson four-momenta. Thus taking

$$M(K^+\to\pi^++\pi^0)=a+bq_k^{\,2}+cq_+^{\,2}+dq_0^{\,2}$$

and

$$M(K_1^{\,0}\to\pi^++\pi^-)=a'+b'q_k^{\,2}+c'q_+^{\,2}+d'q_-^{\,2}$$

($c'=d'$ by CP invariance) our result (13) leads to the determination $a'=a=0$, $b'=b=0$, $c'=d=-c$, so that the result (2) follows immediately. In the treatment of Hara and Nambu the essential difference in the determination of the constants a, b, $\cdots$ comes from the fact that when q_- (for instance) $\to 0$, one must set $q_k=q_+$ so that q_k^2 and q_+^2 are no longer independent variables, as is the case in our work. In Hara and Nambu's work thus, whereas $b=0$, b' is not 0. In fact, one has in this case $a'=a=0$, $b=0$, $c=-d$, $b'+c'=b+c$. On further imposing the condition that in the SU(3) limit $M(K\to 2\pi)=0$ one obtains $b'=-2d$, yielding the result (1). The fact that here b' does not vanish is the reason that leads to the suppression of R.

[8]L. J. Clavelli, Enrico Fermi Institute for Nuclear Studies Report No. EFINS 66-106, University of Chicago, 1967 (unpublished); J. Schechter, Enrico Fermi Institute for Nuclear Studies Report No. EFINS 67-33, University of Chicago, 1967 (unpublished); Y. Hara, Progr. Theoret. Phys. 37, 470 (1967).

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[10]S. Fubini, Nuovo Cimento 43A, 475 (1966); R. F. Dashen and M. Gell-Mann, in Proceedings of the Fourth Coral Gables Conference on Symmetry Principles at High Energy, University of Miami, January 1967, edited by A. Perlmutter and B. Kurşunoğlu (W. H. Freeman & Company, San Francisco, California, 1967).