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[4]See the papers of Ref. 3 for a discussion of representations and notation.

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DETERMINATION OF THE SCATTERING AMPLITUDES FROM POLARIZATION MEASUREMENTS

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A necessity condition is proved for the set of polarization measurements which have to be performed for a full determination of the scattering amplitudes at fixed angles. The special cases of resonance scattering, Regge poles, and potential scattering are discussed.

Introduction and main result.—Scattering processes are completely described by $N = \sum_\nu (2j_\nu + 1)$ scattering amplitudes $R_{m_\nu} = R_{m_1 \cdots m_n}$ which are functions of the dynamical variables. Each m_ν labels the magnetic quantum number of the intrinsic spin j_ν of the νth particle in some suitable coordinate system and n is the number of particles with spin $j \neq 0$ in the reaction. In some cases the number N of independent amplitudes can be reduced by parity conservation, etc.

By measurements, however, linear combinations of bilinear products $R_{m_\nu} R_{m_{\nu'}}{}^*$ are determined. There are N^2 linearly independent measurements at given dynamical variables. These include measurements of the polarization of all incoming and outgoing particles in coincidence, which is in general outside the reach of experimental possibilities.

Because $(R_{m_\nu} R_{m_{\nu'}}{}^*)^* = (R_{m_{\nu'}} R_{m_\nu}{}^*)$, the N^2 possible measurements give N^2 real parameters from which $2N-1$ real parameters of the amplitudes can be determined while one overall phase is undeterminable. Conversely these $2N-1$ parameters uniquely determine the N^2 measurements.

This leads to the following question: what measurements are necessary and which sets of measurements are sufficient to determine all the scattering amplitudes R_{m_ν} up to a common phase?

In the present paper the following necessity statement will be proved[1]: If no phase analysis is performed, the polarization of each particle (with $j \neq 0$) must be measured in coincidence with at least one other polarization, and this in such a way that it is impossible to divide the particles into two sets with no polarization correlation measured between a particle from one and a particle from the other set, unless the missing information according to this criterion can be obtained from the performed measurements by the use of symmetry operations which interchange the role of the particles in the reaction.[2] A continuous family of possible amplitudes will be found if this condition is not met.[3]

A mathematical exception to this statement will be given at the end of the proof and the physical meaning of the exception will be discussed in the last paragraph. The statement holds with or without parity conservation. The proof, however, will be given only without consideration of parity.[1]

Proof.—To prove the above statement we divide the particles with $j \neq 0$ into two sets S_1 and S_2 and show that even if all possible correlation measurements within the two sets are performed there is still a continuous family of solutions for the scattering amplitudes.

Let α and β denote all possible combinations of the magnetic quantum numbers m_i, for $i \in S_1$ and $i \in S_2$, respectively. Then the set of amplitudes R_{m_ν} can be written in form of a (nonsquare) matrix $R = (R_{\alpha\beta})$ which can be interpreted as a mapping R: $E_{S_2} \to E_{S_1}$ of the product space E_{S_2} of the particles in S_2 into the product space E_{S_1} of the particles in S_1.

Now measurements where all particles of S_2 are unpolarized give bilinear products which are averaged over β and the complete determination of H,

$$H_{\alpha\alpha'} = \sum_\beta R_{\alpha\beta} R_{\alpha'\beta}^*, \tag{1}$$

or in form of a matrix

$$H = RR^+, \tag{1'}$$

is the maximum information obtainable from such measurements. Conversely, knowledge of

$$G = R^+R. \tag{2}$$

contains the maximum information obtainable from measurements where the particles of S_1 are unpolarized.

Now assume that R is a simultaneous solution of Eqs. (1) and (2). Then $R' = RU$ with unitary U and $[U, G] = 0$ is also a solution as can easily be seen. A continuous family of such U's exist since G is Hermitian.

It remains to be checked if $R' = RU$ contains other solutions than just $R' = Re^{i\varphi}$. For this we remark that

$$\operatorname{rank}(R) = \operatorname{rank}(G) = \operatorname{rank}(H) \tag{3}$$

and for the kernels K of the mappings[4] we have

$$K(R) = K(G) = K(R') = K(RU). \tag{4}$$

Consequently $K(R)$ and its orthogonal complement $K(R)^\perp$ are invariant under U and therefore U is the direct sum of U^K and $U^{K^\perp}$ where U^K and $U^{K^\perp}$ are automorphisms of $K(R)$ and $K(R)^\perp$, respectively. Now every vector x can be written uniquely as a sum $x = x_1$ with $x_1 \in K(R)$ and $x_2 \in K(R)^\perp$. But $Rx_1 = RUx_1 = 0$ while $Rx_2 \neq RUx_2$ if $x_2 \neq Ux_2$.[5] Since $Rx = R'x$ for all x implies that $R = R'$, we find that U^K does not affect R while different $U^{K^\perp}$ give different R', and $R' = Re^{i\varphi}$ if and only if $U^{K^\perp} = Ie^{i\varphi}$. Since the dimension of $K(R)^\perp$ equals $\operatorname{rank}(R)$ this leads to the following results: (1) $\operatorname{rank}(R) = 0$; this trivial case is equivalent to $R_{m\nu} \equiv 0$. (2) $\operatorname{rank}(R) = 1$; here we have always $U^{K^\perp} = Ie^{i\varphi}$. Since S_1 and S_2 contain particles with $j \neq 0$ this is an "accident." (3) $\operatorname{rank}(R) \geq 2$; then there exist nontrivial $U^{K^\perp}$ and an at least $[\operatorname{rank}(R) - 1]$-dimensional manifold of amplitudes which differ by more than a phase.

Thus the statement of the first paragraph holds with the exception of the cases where $\operatorname{rank}(R) \leq 1$.

Discussion of $\operatorname{rank}(R) = 1$.—We first remark that if $\operatorname{rank}(R) = 1$ the amplitude can be factorized:

$$R_{\alpha\beta} = s_\alpha t_\beta, \tag{5}$$

where the vector $t^* = (t_\beta^*)$ spans (the one-dimensional) $K(R)^\perp$ and $s_\alpha = \sum_\beta R_{\alpha\beta} t_\beta^* [\sum_\beta t_\beta^* t_\beta]^{-1}$. We now discuss some specific examples.

First consider $a + b \to c + d$. If this reaction goes over an intermediate $j = 0$ resonance the amplitude factorizes into production and decay amplitudes according to Eq. (5) with $S_1 = \{c, d\}$ and $S_2 = \{a, b\}$ (i.e., $R_{m_c m_d;\, m_a m_b} = s_{m_c m_d} t_{m_a m_b}$, where s and t are the decay and the production amplitude, respectively). A similar conclusion holds for

$$\begin{array}{l} a + b \to c + d \\ \qquad\quad \lfloor\!\to e + f \end{array}$$

with $S_1 = \{a, b, d\}$ and $S_2 = \{e, f\}$ if c is a $j = 0$ resonance.

In Regge-pole theory the corresponding situation is encountered for the crossed channel if the factorization conjecture[6] for the residues is assumed to hold and if only one Regge pole contributes to the asymptotic scattering amplitude. Here the restriction to spin 0 for the contributing trajectory does not hold since the kinematical factors can be factorized in the asymptotic limit.[7] If the reaction is $a + b \to c + d$ and the trajectory is in the crossed channel $a + \bar{c} \to \bar{b} + d$, Eq. (5) holds with $S_1 = \{a, c\}$ and $S_2 = \{b, d\}$.

A third example is given by potential scattering without spin coupling.

It should be remarked that in general the reaction amplitude cannot be factorized if in one set there is only one particle unless this is coupled to a spin-0 particle.

Finally, remark that the theorem proved applies for the determination of r and s in Eq. (5) independently.

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[1]A more complete analysis will be given elsewhere.

[2]An example of such a symmetry operation is time reversal in elastic scattering which allows interchange of the roles of incoming and outgoing particles. Measurements of correlations between the incoming particles thus already contain the information which could be gained from correlation measurements between the outgoing particles. Other examples are permutation of identical particles, or CP invariance for particle-antiparticle scattering (see also Ref. 1).

[3]Thus, to take just the $2N-1$ easiest measurements can be completely misleading.

[4]Remark that H and G can be interpreted as endomorphisms of E_{S_1} and E_{S_2}, respectively. The kernel of a

mapping is, as usual defined as the set of all vectors which are mapped on the zero vector.

[5]This is well known, otherwise $R(x_2-Ux_2)=0$, i.e., $x_2-Ux_2 \in K(R)$, which is impossible since with x_2 and Ux_2 also their difference is in $K(R)^{\perp}$.

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[7]G. C. Fox and Elliot Leader, Phys. Rev. Letters 18, 628 (1967).

ERRATA

TOTAL AND DIFFERENTIAL CROSS SECTIONS FOR $\pi^- + p \rightarrow \eta + n$ FROM THRESHOLD TO 1300 MeV. W. Bruce Richards, Charles B. Chiu, Richard D. Eandi, A. Carl Helmholz, Robert W. Kenney, Burton J. Moyer, John A. Poirier, Robert J. Cence, Vincent Z. Peterson, Narender K. Sehgal, and Victor J. Stenger [Phys. Rev. Letters 16, 1221 (1966)].

(1) The lower limit of integration in Eq. (2) should read

$$\beta^{-1}\cos(\varphi_{\text{max}}/2).$$

(2) The caption of Table I should be, "Partial η-production cross section, ratio of η to π production, and coefficients of the Legendre-polynomial expansion of the η differential cross section, normalized to the partial-production cross section. Errors given for the coefficients do not include error of normalization."

EXACT RELATION FOR MAGNON THEORIES OF MAGNETISM AND ITS CONSEQUENCES IN CALCULATIONS OF MAGNON SPECIFIC HEATS. R. E. Mills [Phys. Rev. Letters 18, 1189 (1967)].

In Eq. (9), the right-hand side should read $2\delta_{jk}\langle S_{jz}\rangle$. In Eq. (13), the factor $(\omega - g\mu_{\text{B}}H)$ should be replaced by $(\omega + g\mu_{\text{B}}H)$. In Eq. (14), the integrand should read $\rho(q;\omega)[\omega + g\mu_{\text{B}}H + 2zJ\gamma_q/(e^{\beta\omega}-1)]$.