EXCLUSION PRINCIPLE AND THE NEUTRON-ALPHA EQUIVALENT POTENTIAL

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Because of the exclusion principle, the phase/binding energy-equivalent n-α potential in the S state should have a repulsive core or Pauli barrier $V(r\to 0)\to(\hbar^2/2\mu)6/r^2$. A distorted inverse-square barrier wave approximation is used to deduce from experimental phases an entirely repulsive, $l=0$ n-α potential.

Resonating group structure calculations, employing central nucleon-nucleon potentials[1,2] and tensor forces,[3-5] predict n-α elastic scattering phases $\Delta_l(k)$. These show $\Delta_0(0)=\pi$, $\Delta_0(\infty)=0$, with ^{5}He unbound $(n_0=0)$.

A many-body Levinson theorem[6] predicts m_0 redundant bound-state–type solutions to the wave equation (m_0 = number of "excluded" composite bound states). If the potential (part central, part nonlocal) satisfies

$$\int_0^\infty dr\, r^j\, |U(r)|\, dr<\infty,\quad j=1,2, \tag{1}$$

$$\int_0^\infty dr\, r^{j-1}\int_0^\infty dr'\, r'\, |K_0(r,r')|<\infty,\quad j=1,2, \tag{2}$$

then

$$\Delta_0(0)-\Delta_0(\infty)=(n_0+m_0)\pi,\quad \Delta_0(\infty)=0. \tag{3}$$

We require a two-body n-α central potential which is phase/binding energy-equivalent to the above.

Condition (1) gives phases $\eta_0(k)$ satisfying Levinson's theorem:

$$\eta_0(0)-\eta_0(\infty)=n_0\pi,\quad \eta_0(\infty)=0. \tag{4}$$

Since $n_0=0$, one has $\eta_0(0)=0$, requiring $\eta_0(k)=\Delta_0(k)-\pi$; but since $\eta_0(\infty)=-\pi$, condition (1) is broken, invalidating (4). Thus the potential function $U(r)=(2\mu/\hbar^2)V(r)$ has a repulsive core or Pauli barrier $U_{\text{s.c.}}(r\to 0)\to 6/r^2$ (exclusion-principle effect). An intrinsic repulsive core in the nucleon-nucleon potential may be absorbed in cut-off factors.

One form of the Eckart potential possessing this property is

$$U_{\text{s.c.}}(r)=6\lambda^2 e^{-\lambda r}/(1-e^{-\lambda r})^2. \tag{5}$$

Its solutions, regular and irregular, respectively, at $r=0$, are[7]

$$F_0(k,r)=N(k)[-\{4k^2+\lambda^2-3\mu^2(r)\}\sin kr+6k\mu(r)\cos kr],$$
$$G_0(k,r)=N(k)[-\{4k^2+\lambda^2-3\mu^2(r)\}\cos kr-6k\mu(r)\sin kr], \tag{6}$$

where

$$N(k)=\tfrac{1}{2}[(4k^2+\lambda^2)(k^2+\lambda^2)]^{-1/2},$$
$$\mu(r)=-\lambda(1+e^{-\lambda r})/(1-e^{-\lambda r}). \tag{7}$$

We have

$$F_0(k,r\to 0)\approx r^3,\quad F_0(k,r\to\infty)=\sin(kr+\delta_0),$$
$$G_0(k,r\to 0)\approx 1/r^2,\quad G_0(k,r\to\infty)=\cos(kr+\delta_0), \tag{8}$$

the properties for $r\approx 0$ being similar to D waves. Also,

$$k\cot\delta_0(k)\equiv -C_{00}+C_{01}k^2, \tag{9}$$

with

$$\delta_0(0)=0,\quad \delta_0(\infty)=-\pi,$$

and

$$C_{00}=\frac{1}{3}\lambda,\quad C_{01}=\frac{2}{3\lambda},\quad C_{0n}(n\geq 2)\equiv 0. \tag{10}$$

For n-α $l=0$ scattering, $E\leq 8$ MeV phase analysis yields

$$C_{00}\approx 0.3999\ \text{F}^{-1},$$
$$C_{01}\approx 0.6475\ \text{F},\quad C_{02}\approx -0.02987\ \text{F}^3. \tag{11}$$

Fitting $C_{00}=0.3999$ via (10) gives $\lambda=1.1997\ \text{F}^{-1}$, implying $C_{01}=0.5557$, $C_{02}\equiv 0$—an approximate fit to (11).

Note that an infinite square barrier (hard core) of width b has

$$C_{00}=1/b,\quad C_{01}=\tfrac{1}{3}b,\quad C_{02}=b^3/45,\quad\cdots;$$

and fitting $C_{00}=0.3999$ leads to $b\approx 2.5006$ F, with $C_{01}=0.8335$, $C_{02}=0.3475$—a much worse fit than for the Pauli barrier.

The distorted inverse-square barrier wave approximation (DISBWA) method calculates the departure of $U(r)$ from $U_{\text{s.c.}}(r)$ via

$$U(r)=U_{\text{s.c.}}(r)+U_p(r),$$

$$U_p(r)=-\sum_{n=1}^{N} B_n \exp(-\lambda_n r) \tag{12}$$

with $\lambda_n=\lambda_1/n$, say. The scattering wave function satisfies

$$u_0(k,r\to 0)\approx r^3,$$

$$u_0(k,r\to\infty)=v_0(k,r)=\sin(kr+\eta_0), \tag{13}$$

so that

$$u_0(k,r)\approx F_0(k,r)\cos(\eta_0-\delta_0)+g_0(k,r)G_0(k,r)\sin(\eta_0-\delta_0), \tag{14}$$

where $g_0(k,r)$ is a form factor with properties

$$g_0(k,r\to 0)\approx r^5, \quad g_0(k,r\to\infty)=1. \tag{15}$$

A suitable form is

$$g_0(k,r)=(1-e^{-\gamma_0 r})^5\Big[1+e^{-\gamma_0 r}\sum_{\nu=1}^{N}\epsilon_{0\nu}(kr)^{2\nu}\Big], \tag{16}$$

where γ_0 and $\epsilon_{0\nu}$ are adjustable parameters.

On using (12), one finds the integral equation for the phase difference $\eta_0-\delta_0$:

$$\sin(\eta_0-\delta_0)=-\frac{1}{k}\int_0^\infty U_p(r)F_0(k,r)u_0(k,r)dr. \tag{17}$$

By substitution of (14) in (17), we get the DISBWA integral equation for $U_{\text{s.c.}}(r)$, valid for $|\eta_0(k)-\delta_0(k)|<\pi$ $(k>0)$:

$$\tan(\eta_0-\delta_0)=K_{10}/(1-K_{20}), \tag{18}$$

$$K_{10}=-\frac{1}{k}\int_0^\infty U_p(r)F_0^{\,2}(k,r)dr,$$

$$K_{20}=-\frac{1}{k}\int_0^\infty U_p(r)g_0(k,r)F_0(k,r)G_0(k,r)dr. \tag{19}$$

We write

$$K_{10}\approx N^2(k)\sum_{n=0}^{N}(-1)^n\alpha_{0n}k^{2n},$$

$$K_{20}\approx N^2(k)\sum_{m=0}^{N}\beta_{0m}k^{2m}, \tag{20}$$

where α_{0n} and β_{0m} involve weighted moment integrals over $U_p(r)$ and are functions of γ_0 and ϵ_{0n} $(n=1-N)$. Equation (18) reduces to the shape-dependent formula

$$k\cos[\eta_0(k)-\delta_0(k)]\approx\sum_{n=0}^{N}(-1)^{n+1}A_{0n}k^{2n}. \tag{21}$$

The numerical coefficients A_{0n} follow from the shape-dependent formulas

$$k\cot\eta_0(k)\approx\sum_{m=0}^{N}(-1)^{m+1}C_{0m}k^{2m},$$

$$k\cot\delta_0(k)\equiv-\frac{1}{3}\lambda+\frac{2}{3\lambda}k^2, \tag{22}$$

with $A_{0n}(n\geq 2)=A_{0n}(E_{\max})$ and $C_{0m}(m\geq 2)=C_{0m}(E_{\max})$ for $E\leq E_{\max}$. One finds a set of linear relations between coefficients α_{0n} and β_{0n}:

$$\sum_{m=0}^{n}\alpha_{0,n-m}A_{0m}$$

$$=(-1)^n\beta_{0n}-4(\lambda^4\delta_{n0}-5\lambda^2\delta_{m}+4\delta_{n2}), \quad 0\leq n\leq N. \tag{23}$$

This is similar to the standard distorted plane-wave approximation (DPWA) form,[8] except for the last bracketed term replacing $-\delta_{n0}$.

Given γ_0 and ϵ_{0n} $(1<n<N)$ and assuming (12) for $U_p(r)$, Eq. (23) gives $N+1$ simultaneous equations linear in B_n $(n=1-N)$, but nonlinear in λ_1. To find γ_0 and ϵ_{0n}, we expand in powers of k^2 the relation

$$k\cot\eta_0(k)+C_{00}$$

$$=k^2\int_0^\infty[v_0(k,r)v_0(0,r)-u_0(k,r)u_0(0,r)]dr, \tag{24}$$

obtaining relations for C_{0n} $(1\leq n\leq N)$ as functions of γ_0 and (linearly) of ϵ_{0n} $(n=1\cdots N)$. We thus obtain γ_0 and ϵ_{0n} from the experimental C_{0n} $(1\leq n\leq N)$ values.

Numerical calculations show that to fit C_{00} and C_{01} in (11), we require $\lambda<1.1999-E$, $0<E<0.03$. The value nearest to the figure $C_{02}=-0.0297$ of (11) is for $\lambda\approx 1.10$, where $C_{02}(\lambda)$ has a minimum at $C_{02}\approx 1.22$. We take $\lambda=1.10$, $\gamma=0.57166$, and fit C_{02} by using $\epsilon_{01}=-1.2344$, the latter ϵ_{01} value altering $u_0(r)$ from its $\epsilon_{01}=0$ values by only 2-5 % in the interaction region $r<6$ F, say.

Solution of (23) with $N=1$ gives $B_1=0.13703$ F^{-2}, $\lambda_1=0.61544$ F^{-1}, subject to errors of several percent as in DPWA calculations.[8]

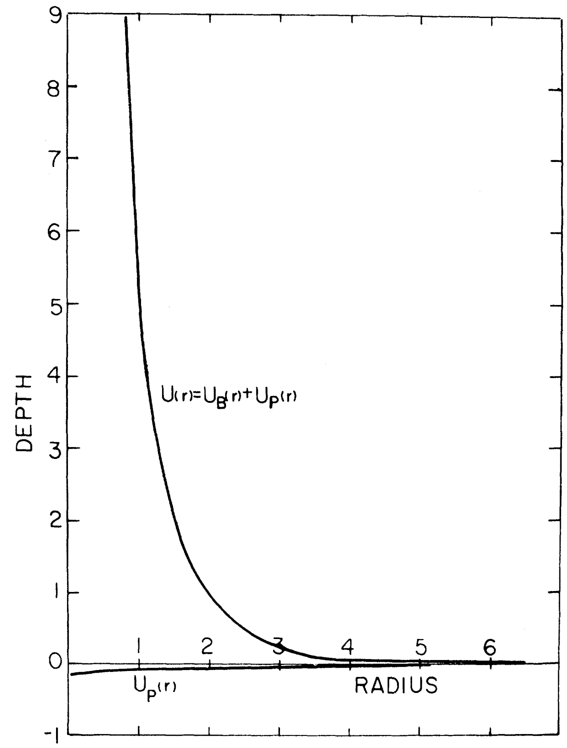

FIG. 1. S-wave scattering potential ($k^2 \times 10^{26}$ cm^{-2}) as function of radius (F). $U(r)$ is the total potential; $U_p(r)$ the nuclear potential component, and $U_{\text{s.c.}}(r)$ (not shown separately) the Pauli barrier.

We get

$$U(r) = 7.26e^{-1.10r}/(1-e^{-1.107})^2 - 0.137\,03e^{-0.615\,44r} \qquad (25)$$

in units of F^{-2} (see Fig. 1). Thus $U_p(r)$ is a very small attractive correction ≈3.4-MeV deep to the Pauli barrier, with range $[U_p(r) = -B_i e^{-2r/b}]$ $b = 3.25$ F, consistent with an optical-well range based on the nucleon radius $r_0 = 1.256$ F.

Figure 2 shows the $P_{1/2}$ and $P_{3/2}$ potentials found in earlier DPWA calculations,[8,9] approximately 83 and 110 MeV deep, with ranges 2.18 and 2.34 F, respectively. The P-state potentials have strengths consistent with an optical-model potential, but the S-state nuclear potential $U_p(r)$ is only a few percent of the strength of an optical well.

The result can be understood if we remember that this is an equivalent two-body problem, involving two neutrons with quantum numbers which would be identical but for one neutron being in an $l=0$ state in the α-particle c.m. system, and the other in an $L=0$ state of the n-α c.m. system. Two such free neutrons would have a zero state probability $|u_0(r)|^2 = 0$, but we treat the α particle as a neutron "tethered" to the α-particle c.m. This neutron has small $L \geqslant 1$ components in the n-α c.m. system, giving rise to the small residual optical-well interaction $U_p(r)$, and to the finite range of $U_{\text{s.c.}}(r)$. But for the $L>1$ components, we would have $\lambda = 0$ and $U_{\text{s.c.}}(r) = 6/r^2$, so that $\eta_0(k) = \pi$, equivalent to $l=2$ and $\eta_2(k) = 0$, or no interaction in the S state.

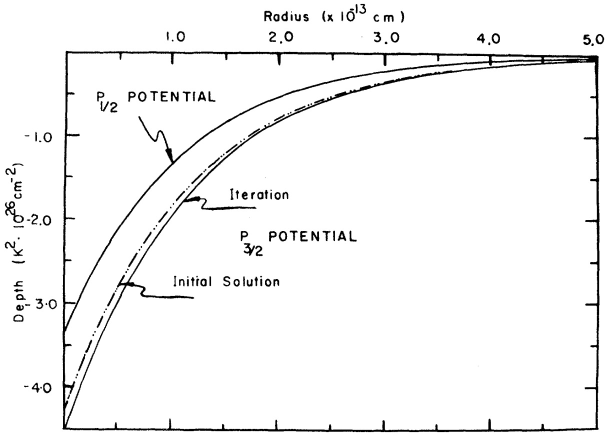

FIG. 2. P-wave scattering potentials as functions of radius. The $P_{1/2}$ potential is compared with the initial and iterated $P_{3/2}$ solution potential. The difference between the $P_{3/2}$ and $P_{1/2}$ potentials is due to spin-orbit effects (energy units are such that $k^2 = 1.0$ F^{-2} corresponds to 25.65 MeV in the c.m. system).

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