TESTS OF NEW π-N SUPERCONVERGENT DISPERSION RELATIONS*

Yu-Chien Liu and S. Okubo
Department of Physics and Astronomy, University of Rochester, Rochester, New York
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A new type of superconvergent sum rules for π-N scattering amplitudes has been derived and compared with the experiments.

In this note we propose a new set of dispersion sum rules for pion-nucleon scattering. We find that they are reasonably well satisfied, thus giving one more evidence of the internal consistency of the dispersion relation.

Our sum rules are essentially superconvergent dispersion relations, with only one essential difference in that they contain integrals of both the real part and the imaginary part of the scattering amplitude. To begin, let us consider the forward pion-nucleon scattering amplitude $C^{(-)}(\nu)=A^{(-)}(\nu)+\nu B^{(-)}(\nu)$. The notation is standard,[1] e.g., $\mathrm{Im}C^{(-)}(\nu)$ is related to the total scattering cross sections of $\pi^{\pm}p$ by the formula

$$\mathrm{Im}C^{(-)}(\nu)=\tfrac{1}{2}(\nu^2-\mu^2)^{1/2}[\sigma_-(\nu)-\sigma_+(\nu)]. \tag{1}$$

Then, let us set

$$F(\nu)=e^{\pi\beta i}(\nu^2-\mu^2)^{-\beta}C^{(-)}(\nu), \tag{2}$$

where μ is the pion mass and β is an arbitrary constant satisfying the condition

$$1>\beta>\tfrac{1}{2}[1+\alpha_\rho(0)] \tag{3}$$

[$\alpha_\rho(0)$ being the value of ρ-meson Regge trajectory $\alpha_\rho(t)$ at $t=0$]. We shall choose the cuts of $(\nu^2-\mu^2)^{-\beta}$ to run from ∞ to μ and from $-\mu$ to $-\infty$, and normalize the function $(\nu^2-\mu^2)^{-\beta}$ such that it assumes a real number on the right-hand cut in the upper-half complex ν plane. In this way $F(\nu)$ is a meromorphic function of ν, with branch cuts stretching over $\infty>\nu\geqslant\mu$ and $-\infty<\nu\leqslant-\mu$, and with poles at $\nu=\pm\nu_B=\mp\mu^2/2m$, where m is the nucleon mass. The additional factor $e^{\pi\beta i}$ has been so chosen that $\mathrm{Im}F(\nu)=0$ in the real interval $-\mu<\nu<\mu$, except for the pole contribution at $\nu=\pm\nu_B$. If we assume the Regge behavior for $C^{(-)}(\nu)$ when ν is asymptotic, we can find that $F(\nu)\sim\nu^{-\Delta}$ $(\nu\to\infty)$ with $\Delta=2\beta-\alpha_\rho(0)>1$, in the light of Eq. (3). From this we have a superconvergent dispersion relation[2] for $F(\nu)$:

$$\int_{-\infty}^{\infty}d\nu\,\mathrm{Im}F(\nu+i\epsilon)=0. \tag{4}$$

Since $C^{(-)}(\nu)$ is a crossing-odd function of ν, this can moreover be rewritten as

$$\int_\mu^\infty\frac{d\nu}{(\nu^2-\mu^2)^\beta}\{\cos(\pi\beta)\,\mathrm{Im}C^{(-)}(\nu)+\sin(\pi\beta)\,\mathrm{Re}C^{(-)}(\nu)\}=-\frac{g^2}{2m}\frac{\pi\nu_B}{(\mu^2-\nu_B^{\,2})^{\beta}}. \tag{5}$$

When $\beta\to1-0$, Eq. (5) reduces to the ordinary dispersion relation

$$\mathrm{Re}C^{(-)}(\mu)=-\frac{\mu}{m}g^2\frac{\nu_B}{\mu^2-\nu_B^{\,2}}+\frac{2\mu}{\pi}\int_\mu^\infty d\nu\frac{1}{\nu^2-\mu^2}\,\mathrm{Im}C^{(-)}(\nu). \tag{6}$$

Similarly, let us consider another new function

$$G(\nu)=e^{\pi\gamma i}(\nu^2-\mu^2)^{-\gamma}[C^{(-)}(\nu)-(\nu/\mu)C^{(-)}(\mu)], \tag{7}$$

where γ is another arbitrary real number, satisfying

$$1<\gamma<\tfrac{3}{2}. \tag{8}$$

By the same token we obtain one more superconvergent dispersion relation

$$\int_{-\infty}^{\infty}d\nu\,\mathrm{Im}G(\nu+i\epsilon)=0, \tag{9}$$

which in turn gives

$$\int_\mu^\infty \frac{d\nu}{(\nu^2-\mu^2)^\gamma}\left\{\cos(\pi\gamma)\,\mathrm{Im}C^{(-)}(\nu)+\sin(\pi\gamma)\left[\mathrm{Re}C^{(-)}(\nu)-\frac{\nu}{\mu}\,\mathrm{Re}C^{(-)}(\mu)\right]\right\}=\frac{g^2}{2m}\frac{\pi\nu_B}{(\mu^2-\nu_B^2)^\gamma}. \tag{10}$$

When $\gamma\to 1+0$, this equation reduces again to the ordinary dispersion sum rule Eq. (6), while as $\gamma \to \frac{3}{2}-0$, it gives the following relation essentially equivalent to the one originally derived by Gilbert[3] some years ago:

$$\lim_{\nu\to\mu+0}\left[\frac{\mathrm{Im}C^{(-)}(\nu)}{(\nu^2-\mu^2)^{1/2}}\right]=\frac{\mu}{m}g^2\frac{\nu_B}{(\mu^2-\nu_B^2)^{3/2}}-\frac{2\mu}{\pi}\int_\mu^\infty\frac{d\nu}{(\nu^2-\mu^2)^{3/2}}\left[\mathrm{Re}C^{(-)}(\nu)-\frac{\nu}{\mu}\,\mathrm{Re}C^{(-)}(\mu)\right]. \tag{11}$$

In the following we shall test our sum rules Eqs. (5) and (10). The final results are summarized in Tables I and II. In the evaluation of $C^{(-)}(\nu)$, we used the scattering length approximation,[1,3] i.e., we retain only the relation $k\cot\delta = 1/a$ and keep only the S wave, for the incident pion momentum less than 0.02 GeV/c. The adopted scattering lengths[4] are $a_1 = 0.178\ \mu^{-1}$ and $a_3 = -0.107\ \mu^{-1}$. [In this connection, the old values of $a_1 = 0.171\ \mu^{-1}$ and $a_3 = -0.088\ \mu^{-1}$ given by Hamilton and Woolcock[5] yield a worse fit for the sum rule Eq. (10).] From 0.02 to 8.00 GeV/c, both $\mathrm{Im}C^{(-)}(\nu)$ and $\mathrm{Re}C^{(-)}(\nu)$ are supplied by Höhler and Strauss.[6] Their $\mathrm{Re}C^{(-)}(\nu)$'s have been evaluated from the ordinary dispersion relation, and in most cases are consistent with experimental values. For the region from 8 to 24 GeV/c, Lindenbaum's new data[7] are used. From 24 GeV/c up to infinity, the Regge-pole model may be used. However, even up to 24 GeV/c the assumption that only ρ trajectory is exchanged is not so satisfactory for the actual experiment (Lindenbaum found a fit with $\alpha_\rho(0) = 0.7$, a bit larger), as well as for the sum rule Eq. (5), although the contributions are not so large. As a consequence, we still use Lindenbaum's semiexperimental formula[7] to extrapolate to infinity.

As we see from Tables I and II, the agreement is in general satisfactory except for $\beta \approx 0.80$ in Eq. (5) and $\gamma \approx 1.50$ in Eq. (10), where both sides of the equations differ nearly 10% from each other. It is likely that these errors are due to our still-imperfect knowledge of $\mathrm{Re}C^{(-)}(\nu)$. Actually, Gilbert[3] had tried to determine the scattering lengths from Eq. (6) (which gives $a_1 - a_3$) and from Eq. (11) [which, combined with Eq. (6), gives $a_1 + 2a_3$]. Although Eq. (11) is less dependent on the high-energy behavior of $C^{(-)}(\nu)$, the integral has an appreciable contribution in the region $\nu \approx \mu$, where the accuracy of the present $\mathrm{Re}C^{(-)}(\nu)$ is by no means good enough to offer a better determination. We hope that the as-yet-inaccurate data on the real part of the scattering amplitude may be remedied in the future.

Table I. Test of Eq. (5) in the context. In the right-hand side we used $f^2/4\pi = (\mu/2m)^2 g^2/4\pi = 0.081$.

β	Left-hand side	Right-hand side
1.00	3.2	3.2
0.98	3.2	3.2
0.94	3.2	3.2
0.90	3.2	3.2
0.86	3.3	3.2
0.84	3.3	3.2
0.82	3.4	3.2
0.80	3.5	3.2

Table II. Test of Eq. (10) in the context. Notations same as in Table I.

γ	Left-hand side	Right-hand side
1.50	3.0	3.2
1.46	3.0	3.2
1.40	3.1	3.2
1.34	3.1	3.2
1.28	3.2	3.2
1.22	3.2	3.2
1.16	3.2	3.2
1.10	3.2	3.2
1.04	3.2	3.2
1.00	3.2	3.2

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[1]We have followed the notation given in the book by

K. Nishijima, Fundamental Particles (W. A. Benjamin, Inc., New York, 1964).
[2]V. de Alfaro et al., Phys. Letters 21, 576 (1966).
[3]W. Gilbert, Phys. Rev. 108, 1078 (1957).
[4]S. Gasiorowicz, Elementary Particle Physics (John Wiley & Sons, Inc, New York 1966).
[5]J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. 35, 737 (1963).
[6]G. Höhler and R. Strauss, private communication.
[7]S. J. Lindenbaum, to be published.

$Z=0$ AND COMPOSITENESS

Hugh Osborn
Department of Physics, University College London, London, England
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A distinction is drawn between the divergence of the canonical commutator for composite fields and the condition $Z=0$ for equivalence of an elementary field theory to a composite theory.

Fried and Jin[1] have given a short but rigorous proof of $Z_C=0$ (Z_C being the composite-field renormalization constant) as a necessary condition for composite fields. They evaluated the vacuum expectation value of the canonical equal-times commutator of the composite field constructed from the local product of elementary fields and normalized by the vacuum to composite-particle matrix element being unity, apart from kinematic factors. The condition $1/Z_C=\infty$ then results from the basic divergences of the underlying elementary field theory. Conventional discussion of the $Z=0$ rule for the equivalence of an elementary particle to a corresponding composite particle disregards divergence problems.[2] We endeavor to clarify the situation, and show how $Z=0$ still follows in a nondivergent theory, where it is possible that $Z_C\neq 0$.

Let $\varphi(x)$ be an elementary field (renormalized; only renormalized quantities are considered here) governed by the field equation

$$Z(\Box+\mu^2)\varphi(x)=Z\delta\mu^2\varphi(x)-Z_1GJ(x). \tag{1}$$

$J(x)$ is independent of $\varphi(x)$. If $|p\rangle$ is the one-particle state of mass μ, the normalization is

$$\langle 0|\varphi(0)|p\rangle(2p_0)^{1/2}=\langle 0|\varphi_C(0)|p\rangle(2p_0)^{1/2}=1, \tag{2}$$

$$\varphi_C(x)=(Z_1G/Z\delta\mu^2)J(x). \tag{3}$$

In the limit $Z\to 0$ we expect $\varphi(x)\to\varphi_C(x)$, where $\varphi_C(x)$ is a composite field with no independent degrees of freedom. Both φ and φ_C are local scalar fields and their propagators satisfy the usual Lehmann representation, with poles of unit residue at μ^2. From (1) and (3), the relation between them is

$$\Delta_F'(k^2)=\Delta_{F_C}'(k^2)\left[1+\frac{Z}{Z\delta\mu^2}(k^2-\mu^2)\right]^{-2}+\frac{1}{Z\delta\mu^2}\left[1+\frac{Z}{Z\delta\mu^2}(k^2-\mu^2)\right]^{-1}. \tag{4}$$

Z and $Z\delta\mu^2$ are independent,[3] so[4]

$$\Delta_F'(k^2)_{Z=0}=\Delta_{F_C}'(k^2)+1/Z\delta\mu^2. \tag{5}$$

In a nondivergent theory with $Z\delta\mu^2$ finite and

$$\Delta_{F_C}'(k^2)_{k^2\to\infty}\simeq 1/Z_Ck^2,$$

the right-hand side of (5) is asymptotically a constant, and therefore obviously corresponds to $Z=0$ in $\Delta_F'(k^2)$ on the left-hand side. Only if $Z\delta\mu^2$ diverges do $\Delta_F'(k^2)_{Z=0}=\Delta_{F_C}'(k^2)$ and the divergence of $1/Z_C$, which Fried and Jin considered, prove that $Z=0$.

In general, Z and Z_C are different. The special case in Ref. 1 was given by considering the composite field

$$\varphi_C(x)=:\varphi^2(x):/\lambda, \tag{6}$$

$$\lambda=\langle 0|\varphi^2(0)|p\rangle(2p_0)^{1/2}\lim_{Z\to 0}\frac{Z\delta\mu^2}{Z_1G}, \tag{7}$$

taking $J(x)=:\varphi^2(x):$.

From (4) it is easy to show that

$$\Delta_{F_C}'(k^2)=\Delta_F'(k^2)Z(k^2-\mu^2+\delta\mu^2)\frac{1}{(Z\delta\mu^2)^2}\Pi(k^2), \tag{8}$$

where we have used $\Delta_F'(k^2)=[Z(k^2-\mu^2+\delta\mu^2)-\Pi(k^2)]^{-1}$. From (8), whether or not Z is 0, we have

$$\frac{1}{Z_C}=\frac{1}{(Z\delta\mu^2)^2}\lim_{k^2\to\infty}k^2\Pi(k^2). \tag{9}$$