EXPANSION APPROACH TO SCATTERING

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We describe an expansion method for calculating scattering phase shifts which avoids the difficulties of the variational methods of Hulthén[1] and Kohn.[2] Demkov and Shepelenko[3] have pointed out the algebraic source of these difficulties, and they have shown how the Kohn and Hulthén methods implicitly eliminate an equation to restore consistency. These methods have also been examined elsewhere.[4,5] A third variational method, due to Schwinger,[6] appears to be more difficult to use, although Schwartz[7] has indicated a way to apply Schwinger's method for potential scattering, and Lippmann[8] has recently converted the Schwinger principle into a form similar to those of Hulthén and Kohn. Other approaches are those of Schlessinger and Schwartz,[9] and of Spruch, Rosenberg, Hahn, and O'Malley.[10] The former proceeds by analytic continuation from bound-state regions, while the latter simplifies calculations for compound targets.

Ours is an approach which seeks to use to advantage the singularity properties that adversely affect the Kohn method. As it stands, it is applicable to compound targets, and it appears extendable to multichannel processes. Consider a single-channel process with Hamiltonian H and asymptotic solutions $\psi_1(E)$ and $\psi_2(E)$ at energy E. The stationary-state wave function Ψ is then written $\Psi = a_1\psi_1 + a_2\psi_2 + \Phi$, thereby defining Φ, and the Schrödinger equation yields

$$a_1(H-E)\psi_1(E) + a_2(H-E)\psi_2(E) + (H-E)\Phi = 0. \quad (1)$$

We assume H to be such that at large distances the incident particle sees a short-range potential, so that $(H-E)\psi_1$, $(H-E)\psi_2$, $(H-E)\Phi$, and Φ are all quadratically integrable. The solution of Eq. (1) for Φ then should be expected to be smoothly approximable with bound-state functions.

Introduce a set of bound-state functions χ_i, $i = 1, \cdots, n$, to be used for the approximation of Φ. The role of these functions is most clearly seen if they are transformed to a basis on which H is diagonal within the subspace spanned by the χ_i. That is, we construct and solve the finite matrix equation $(\tilde{H}-\lambda\tilde{S})\,\tilde{c} = 0$, where $H_{ij} = \langle\chi_i H\chi_j\rangle$ and $S_{ij} = \langle\chi_i\chi_j\rangle$. The solution $\tilde{c}_\mu$, corresponding to eigenvalue λ_μ, defines a function $\varphi_\mu = \sum_\nu c_{\nu\mu}\chi_\nu$.

Now, suppose that the functions φ_μ are capable of giving a good representation of Φ at an energy E. Then Eq. (1) should be nearly satisfied. The condition we shall impose here is that the left side of Eq. (1) have no component in the subspace spanned by the φ_μ. Because of the bounded nature of all terms of Eq. (1), this condition should produce convergence to a well-defined solution as the φ_μ approach a complete set of quadratically integrabe functions.

The essence of our method is to recognize that the above-described condition is easy to impose if E is chosen to be an eigenvalue λ_μ of the finite probelm which we used to define the φ_μ. For $E = \lambda_\mu$, the coefficient of φ_μ on the left of Eq. (1) is

$$a_1\langle\varphi_\mu|H-\lambda_\mu|\psi_1(\lambda_\mu)\rangle + a_2\langle\varphi_\mu|H-\lambda_\mu|\psi_2(\lambda_\mu)\rangle.$$

Equating this to 0, we obtain the ratio a_2/a_1, from which we may deduce a phase shift. The whole idea is to use our knowledge of the behavior of H on the bound-state functions to create a set of conditions optimum for determining a_2/a_1.

In practice, the method involves four steps: (1) Choose a set of χ_i and diagonalize H; (2) pick an eigenvalue appropriate to a scattering solution; (3) define ψ_1 and ψ_2 at this energy; and (4) solve for a_2/a_1. By varying the set of χ_i, normally through parametric adjustments, almost any eigenvalue can be reached. Often a single set of χ_i will yield several eigenvalues.

The method yields results which are not changed by the addition to ψ_1 or ψ_2 of any linear combination of the χ_i. This means that whenever the χ_i are actually sufficient to define a good approximate Φ, the phase shifts will not depend upon arbitrariness in the choice of ψ_1 or ψ_2. As already noted, the above procedures apply to compound as well as simple targets. It is of course necessary to take proper account of the spin and statistics in systems containing identical particles.

The usefulness of this new method is determined by the results it gives, as the physics

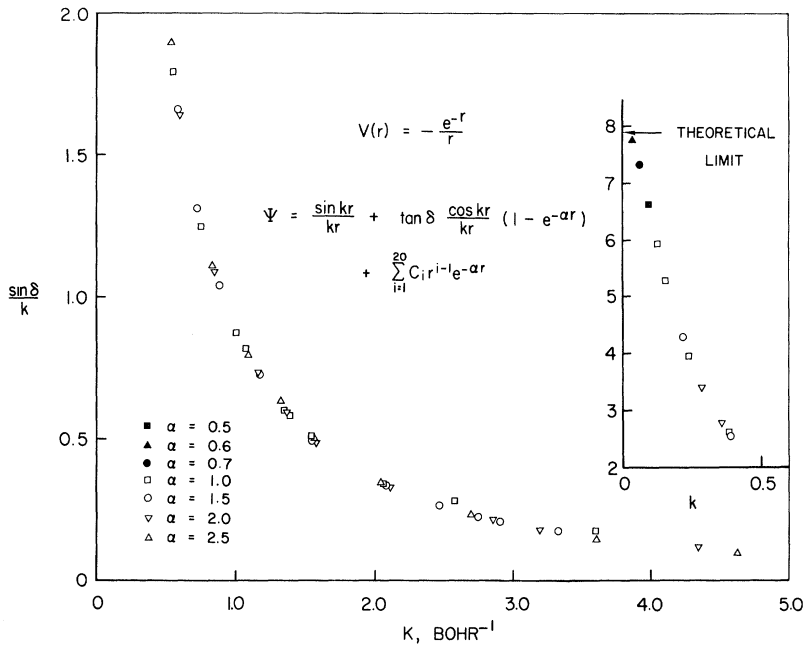

FIG. 1. S-wave phase shifts for an attractive Yukawa potential, based on 20 short-range functions.

is completely contained in the Schrödinger equation, whose validity is not in question here. As a preliminary example, we report on the S-wave scattering by an attractive Yukawa potential. We take $H=(2r)^{-1}(d^2/dr^2)r-r^{-1}e^{-r}$, $\psi_1=(kr)^{-1}\sin kr$, $\psi_2=(kr)^{-1}(1-e^{-\alpha r})\cos kr$, $\chi_i=r^{i-1}e^{-\alpha r}$, $i=1,\cdots,n$. Here $k=(2E)^{1/2}$, and α is a nonlinear parameter to which we assigned various values. With this choice of ψ_1 and ψ_2 the phase shift is $\delta=\tan^{-1}(a_2/a_1)$. Note that the Hamiltonian H supports one bound state.[11] For large expansion lengths n, we obtain results which, except for large k, are completely independent of α. A plot for $n=0$ is shown in Fig. 1. This curve gives a good definition of the scattering over the entire range from 0 to several hundred eV and, incidentally, was produced in a few seconds of computer time. More interesting is the behavior at shorter expansion lengths. We found the results to converge smoothly as a function of n for all α values we tried, with somewhat more rapid convergence near $\alpha=2.5$ than near $\alpha=1.0$. At $\alpha=2.5$, all expansion lengths from $n=2$ yield phase shifts whose deviation from the curve of Fig. 1 is not visually evident. The situation at $\alpha=1.0$ is shown in Fig. 2. Even at this far-from-optimum α value, the results are quite good.

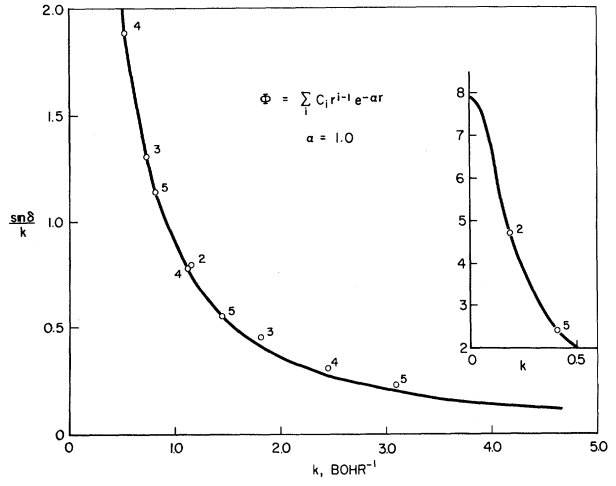

FIG. 2. Convergence of expansion solutions for phase shifts. Solid curve is defined by the fully converged results of Fig. 1. Numbers adjacent to the points indicate the number of short-range functions used.

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NONINVARIANCE GROUP BY BOOTSTRAP AND REGGE BEHAVIOR*

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By means of Regge theory and bootstraps, we motivate a chiral SU(2)⊗SU(2) noninvariance algebra of the meson isobar coupling strengths. We discuss the equivalence of these approaches and compare our algebra with the previously postulated algebras of the meson isobar system.

In the last few years, there has been a great deal of activity concerning the development of relationships between the masses and the coupling constants of the strongly interacting particles. These relationships, called sum rules, are generally the manifestation of an underlying noninvariance algebra. An example of this method is the strong coupling model of Cook, Goebel, and Sakita[1] and its extension by Kuriyan and Sudarshan.[2] In these cases, the basic algebraic structure, which is larger than the underlying symmetry algebra, is, although motivated by a field-theory model, essentially postulated. Most of the sum rules derived in recent years have been motivated by the Regge hypothesis[3] or bootstraps.[4] The purpose of this paper is to show in a specific example how both the Regge and bootstrap hypothesis can be used to motivate the existence of an underlying algebra directly. This derivation of the same algebra by the two methods has also supplied us with an insight into the relationship between them.

The motivation of our algebra by the Regge hypothesis is based on the observation that the forward-scattering amplitude, with isospin one in the t channel, is dominated by the ρ trajectory. Calling this amplitude $M^{(-)}(\nu)$ and assuming that the background integral can be moved to the left without picking up additional contributions, we have that the amplitude $f^{(-)}(\nu) \equiv M^{(-)}(\nu) - R^{(-)}(\nu)$, where $R^{(-)}(\nu)$ is the usual ρ Regge-pole contribution, satisfies a superconvergence relation.[5] We write this relationship as

$$\int_0^\kappa \mathrm{Im}M^{(-)}(\nu)d\nu = \int_0^\kappa \mathrm{Im}R^{(-)}(\nu)d\nu = \frac{b(0)}{\alpha_\rho(0)+1}\kappa^{\alpha_\rho(0)+1}, \quad (1)$$

where we have assumed that there exists a κ such that

$$\int_\kappa^\infty [\mathrm{Im}M^{(-)}(\nu) - \mathrm{Im}R^{(-)}(\nu)]d\nu = 0. \quad (2)$$

We relate this amplitude to the isospin amplitudes by observing that $M^{(-)}(\nu)$ corresponds to the terms odd under pion-isospin exchange:

$$M_{ij}^{\alpha\beta} - M_{ij}^{\beta\alpha} = [\tau^\alpha, \tau^\beta]_{ij} M^{(-)} = i\epsilon^{\alpha\beta\gamma}\tau_{ij}^{\gamma} M^{(-)}, \quad (3)$$

where α and β designate the pion isospin labels and i and j the nucleon labels. Inserting this relationship into Eq. (1) and saturating with single isobar states, we have

$$\sum_n (\varphi_{in}^{\alpha}\varphi_{nj}^{\beta} - \varphi_{in}^{\beta}\varphi_{nj}^{\alpha}) = [\varphi^\alpha, \varphi^\beta]_{ij} = i\epsilon^{\alpha\beta\gamma} I_{ij}^{\gamma} C, \quad (4)$$

where

$$C = b_{NN\rho\pi\pi}(0)\frac{\kappa^{\alpha_\rho(0)+1}}{\alpha_\rho(0)+1}$$