[6]R. J. Tayler, Atomic Energy Establishment Report No. T/R 2263, 1957 (unpublished).

[7]E. S. Weibel, Phys. Fluids 3, 946 (1960). In this paper, aware of the difficulties of the fluid model, the author uses a collisionless model. The results for the ideal fluid model can be found in J. W. Miles, Phys. Fluids 7, 223 (1964).

[8]A list of relevant Russian references can be found in N. A. Bobyrev and O. I. Fedyanin, Zh. Tekh. Fiz. 33, 1187 (1963) [translation: Soviet Phys.—Tech. Phys. 8, 887 (1964)].

[9]R. Campbell, Théorie générale de l'équation de Mathieu (Masson Ed., Paris, France, 1955), p. 190. See also N. W. MacLachlan, Theory and Application of Mathieu Functions (Oxford Univ. Press, Oxford, England, 1947).

[10]F. Troyon, Bull. Am. Phys. Soc. 11, 450 (1966), and to be published.

EVOLUTION OF A NONLINEAR ION ACOUSTIC WAVE*

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We report here what is believed to be the first prediction of the evolution of a nonlinear ion acoustic wave in a Vlasov plasma with the electron temperature T_e much larger than the ion temperature T_i. Considerable work,[1,2] both analytical and experimental, has already been carried out for waves of small amplitude.

The principal result is that we may show close mathematical similarity with the description of a nonlinear sound wave in a neutral gas. Specifically, we may use the well-known Riemann solution[3] for simple waves to prove that large-amplitude compressive ion acoustic pulses steepen and develop discontinuities.

Such behavior often occurs in other situations in which the corresponding linear wave motions are of constant phase velocity.[4] For example, the tendency of compressive hydromagnetic pulses, propagating along a dc magnetic field, to steepen in the nonlinear regime has been demonstrated both theoretically[5] and experimentally.[6] Nonlinear electron plasma oscillations, on the other hand, exhibit no such tendency.

Ideally, the calculation should proceed from the Vlasov equations for the electrons and ions, a formidable task. We proceed instead from a two-fluid model which can, to a certain extent, be justified from a Vlasov description and which correctly predicts the corresponding linearized wave motions.

We ignore the ion temperature and consider, for simplicity, only one-dimensional motions. The ion and electron dynamics are given exactly by the equations.

$$\frac{\partial u_i}{\partial t}+u_i\frac{\partial u_i}{\partial x}=\frac{e}{m_i}E \tag{1}$$

and

$$\frac{\partial u_e}{\partial t}+u_e\frac{\partial u_e}{\partial x}=-\frac{e}{m_e}E-\frac{1}{n_e m_e}\frac{\partial P_e}{\partial x}. \tag{2}$$

The symbols u, n, m, and P stand for fluid velocity, number density, particle mass, and pressure, respectively, and the subscripts i and e stand for ions and electrons. E is the electric field, and the electronic charge is $-e$.

The two fluids each obey an equation of continuity $\partial n_j/\partial t+\partial(n_j u_j)/\partial x=0$, $j=i$ or e. The system is so far exact, but not closed. What is missing is an expression for the electron pressure. We achieve closure by assuming that the electrons respond adiabatically to changes in the electrostatic potential φ $[E=-\partial\varphi/\partial x]$. That is, starting from a Maxwellian distribution, the electron distribution function remains proportional to

$$\exp\{-(\tfrac{1}{2}m_e v^2-e\varphi)/KT_e\}.$$

This is a good approximation for potentials φ which change slowly on the time scale of the electron plasma frequency[7]; and for potentials which vary on the length and time scales of ionic sound waves, it solves the electron Vlasov equation up to terms of the order of the ratio of the ion-acoustic speed, $(KT_e/m_i)^{1/2}$, to the electron thermal speed. It follows immediately that the electron pressure obeys the equation of state

$$\left(\frac{\partial}{\partial t}+u_e\frac{\partial}{\partial x}\right)\left(\frac{P_e}{n_e^{\gamma_e}}\right)=0,$$

with $\gamma_e=1$.

For φ's which change slowly compared with the electron plasma frequency, the approximation of quasi charge neutrality may be made: $|n_e - n_i|/|n_e| \ll 1$, or $n_e \cong n_i = n$, say. Using the continuity equation and imposing the boundary condition that there be no net current at $x = \pm\infty$, one can show that $u_e \cong u_i = u$, say.

Noting that $T_i/T_e \ll 1$, and dropping terms of $O(m_e/m_i)$ relative to those retained, Eq. (2) reduces to

$$0 \cong -\frac{eE}{m_e} - \frac{1}{nm_e}\frac{\partial P_e}{\partial x} \qquad (3)$$

and Eq. (1) becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} \cong -\frac{1}{m_i n}\frac{\partial P_e}{\partial x}, \qquad (4)$$

with the equation of state now being

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\left(\frac{P_e}{n^{\gamma_e}}\right) = 0. \qquad (5)$$

Equations (4) and (5) and the equation of continuity are now just the Euler equations for an ideal fluid, and considerable information exists concerning their solutions. The Riemann solution for simple waves, specifically, is well known[3] and need not be reproduced here. The most noteworthy property of the solution is that any compressive pulse will steepen as it travels, and develop a vertical tangent in a plot of u vs x in a time

$$t_s = 2/(\gamma_e + 1)|\partial u/\partial x|_{\max}, \qquad (6)$$

where $|\partial u/\partial x|_{\max}$ is to be evaluated at $t = 0$.

Whether the wave will develop a shock (as in sound waves in an ordinary gas) or will break (as in water waves) depends upon processes which are omitted in Eqs. (4) and (5). In particular, the quasi charge neutrality will break down when the pulse has steepened enough so that φ changes over an electron Debye length. The electron Debye length may be conjectured to be the length scale of whatever final state is reached after $t = t_s$. The dimensionless parameter which must be $\gtrsim O(1)$ in order to see the nonlinearities clearly is $|e\varphi/KT_e|$. In the wave trains excited in previous experiments, this number has characteristically been $\lesssim 10^{-2}$. The principal experimental limitation in seeing the above phenomenon will apparently be getting stronger coupling between the exciting grids and the ion acoustic wave. Potentials of the order of several volts have been applied in the past but have remained largely confined to the sheath region around the grids. It may be remarked that it is possible that the above effect has been involved in measurements on Q machines.[8]

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[1]B. D. Fried and R. W. Gould, Phys. Fluids 4, 139 (1961); R. W. Gould, Phys. Rev. 136, A991 (1964).

[2]See, for example, I. Alexeff and R. V. Neidigh, Phys. Rev. 129, 516 (1963); W. D. Jones and I. Alexeff, in Proceedings of the Seventh International Conference on Ionization Phenomena in Gases, Belgrade, 1965 (Gradjevinska Knjiga Publishing House, Beograd, Yugoslavia, 1966), Vol. II, p. 322.

[3]See, for example, L. D. Landau and E. M. Lifshitz, Fluid Mechanics (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1959), pp. 366-386.

[4]R. Z. Sagdeev and A. A. Galeev, International Atomic Energy Agency Report No. IC/66/64, 1966 (unpublished).

[5]D. Montgomery, Phys. Rev. Letters 2, 36 (1959).

[6]F. I. Boley and P. R. Forman, Phys. Rev. Letters 12, 385 (1964); I. G. Brown and C. N. Watson-Munro, Plasma Phys. 9, 43 (1967).

[7]I. B. Bernstein and S. K. Trehan, Nucl. Fusion 1, 3 (1960).

[8]H. K. Andersen, N. D'Angelo, P. Michelsen, and P. Nielsen, Phys. Rev. Letters 19, 149 (1967).