[6]G. Domokos and P. Suranyi, Nucl. Phys. 54, 529 (1964).

[7]M. Toller, University of Rome Nota Interna 76 and 84, 1965 (unpublished).

[8]D. Z. Freedman and J. M. Wang, Phys. Rev. (to be published).

[9]To give the energy dependence of the $n+p \to p+n$ forward peak.

[10]R. F. Sawyer, to be published.

[11]Spin nonflip contributions to forward $\pi+N \to \rho+N$ can come from the exchange of an A_1 conspiracy (class II, see Ref. 10). There is some evidence that in the reactions $\pi^- + p \to \rho^0 + n$ and $\pi^- + N \to \rho^0 + (N + n\pi)$ at lab momentum 18 BeV/c, spin-nonflip dominates for the smallest momentum transfers. I am grateful to Professor D. Caldwell for showing me an analysis of his data, a preliminary account of which appeared in Jones et al., Phys. Letters 21, 590 (1966).

[12]The combination $\pi\omega$ has of course been suggested many times as a prime ingredient of a dynamical ρ meson. However the 1^- member of our conspiracy cannot be a ρ (since it becomes uncoupled from any spin-zero plus spin-zero system at zero energy). It would no doubt mix with the ρ, except at zero energy.

[13]A. H. Rosenfeld et al., Rev. Mod. Phys. 39, 1 (1967).

[14]This notation for quantum numbers may be somewhat unconventional. The given dependence of G on J is for the nucleon-antinucleon system. In $\bar{N}+N \to \bar{N}+N$ there is no connection between charge conjugation and signature. (The st spectral region remains the st spectral region under the operation C.) But for the process $\bar{N}+N \to$ two different bosons, charge conjugation interchanges the st and su spectral regions in such a way that the intrinsic G parity of the trajectory, for purposes of coupling to mesons, is a signature-dependent constant along the trajectory. The way to remember the results is, of course, to demand that our $\bar{N}N$ G parity agree with G(meson) at the right-signature integers.

SUM RULES FOR THE SPECTRAL FUNCTIONS OF SU(3) ⊗ SU(3)*

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The spectral-function sum rules are given for a general Lie algebra, and then applied to the vector and axial-vector mesons coupled to the currents of SU(3) ⊗ SU(3).

The spectral-function sum rules[1] for chiral SU(2) ⊗ SU(2) have been used to relate the ρ and $A1$ masses, and to calculate the π^+-π^0 mass difference.[2] In this note we shall state the corresponding sum rules for a general Lie algebra, and will then show that the sum rules for the SU(3) ⊗ SU(3) spectral functions can be used to make sense out of the masses and couplings of the observed vector and axial-vector mesons.

We will first consider a general multiplet of currents which satisfy the commutation relations

$$[J_\alpha{}^0(\vec{x};t), J_\beta{}^\nu(\vec{y},t)] = -iC_{\alpha\beta\gamma}\delta^3(\vec{x}-\vec{y})J_\gamma{}^\nu(\vec{x},t)+S.T., \tag{1}$$

where $C_{\alpha\beta\gamma}$ is the structure constant of a simple Lie algebra,[3] and $S.T.$ denotes a Schwinger term.[4] The spectral-function sum rules are

$$\int\rho_{\alpha\beta}{}^{(1)}(\mu^2)\mu^{-2}d\mu^2+\int\rho_{\alpha\beta}{}^{(0)}(\mu^2)d\mu^2=S\delta_{\alpha\beta}, \tag{2}$$

$$\int\rho_{\alpha\beta}{}^{(1)}(\mu^2)d\mu^2=Z\delta_{\alpha\beta}, \tag{3}$$

with S and Z unknown constants, and $\rho_{\alpha\beta}^{(j)}(\mu^2)$ the spin-j spectral functions:

$$\langle J_\alpha^{\ \mu}(x)J_\beta^{\ \nu}(0)\rangle_0 \equiv (2\pi)^{-3}\int d^4p\,\theta(p^0)\,e^{ip\cdot x}\{[g^{\mu\nu}-p^\mu p^\nu/p^2]\rho_{\alpha\beta}^{(1)}(-p^2)+p^\mu p^\nu\rho_{\alpha\beta}^{(0)}(-p^2)\}. \tag{4}$$

No assumption of current conservation or partial conservation is needed to obtain these results.

The simplest way of deriving (2) and (3) is to use the detailed commutation relations provided by the algebra of gauge fields.[5] Within the context of the usual algebra of currents, it is necessary to make a separate assumption that Schwinger terms in Eq. (1) are c numbers. We can then derive Eq. (2) by using (1) in the Jacobi identity[6] for $J_\alpha^{\ 0}$, $J_\beta^{\ 0}$, and $J_\gamma^{\ i}$, and taking the Fourier transform of its vacuum expectation value. [The c-number Schwinger term does not contribute to the inner commutator of the Jacobi identity.] The resulting equation tells us that the Schwinger term must be linear in momentum and must commute with the generators $(T_\gamma)_{\alpha\beta}=C_{\alpha\gamma\beta}$ of the adjoint representation. Schur's Lemma requires that the Schwinger term be proportional to a Kronecker delta, and the integral representation[7] of the Schwinger term then yields Eq. (2). The second sum rule, Eq. (3), can be derived from assumptions about the high-momentum behavior of the current in precisely the same way as in Ref. 1, but the derivation from the algebra of gauge fields[5] is much more convincing.[8]

We now turn to the special case of SU(3) ⊗ SU(3). Each generator of this algebra carries a unique set of observed quantum numbers (isospin, hypercharge, and parity), so that the $\rho_{\alpha\beta}^{(j)}(\mu^2)$ are diagonal in α and β. Assuming the functions $\rho_{\alpha\alpha}^{(1)}(\mu^2)$ to be dominated by an octet of vector mesons and an octet of axial-vector mesons[9] as shown in Table I, we find from Eqs. (2) and (3) that

$$Zm_\alpha^{-2}+\int\rho_{\alpha\alpha}^{(0)}(\mu^2)d\mu^2=S \text{ (not summed)}, \tag{5}$$

where m_α is the mass of the spin-one meson coupled to $J_\alpha^{\ \nu}$.

We immediately conclude from Eq. (5) that all vector mesons coupled to the currents of good quantum numbers must have equal mass, since in this case $\rho_{\alpha\alpha}^{(0)}=0$. For SU(3) ⊗ SU(3) the good quantum numbers are isospin and hypercharge; so our first prediction is that

$$m_\rho = m_\omega = (Z/S)^{1/2}. \tag{6}$$

Experimentally, $m_\omega/m_\rho \simeq 1.03$. We would claim this as an unqualified success for our approach, but unfortunately we obtained (6) only by leaving the φ meson out of the sum rules. It is a mystery to us why this should work, but we shall see that the same mystery also appears for the axial-vector mesons; so perhaps there is some systematic reason why some particles have to be left out of these sum rules.

The other vector and axial-vector mesons couple to currents that are not conserved; so Eq. (5) tells us only that $m_\alpha > m_\rho$. However, if we assume that the divergences of the currents are dominated by a set of spin-zero mesons $|n,p\rangle$, then Eq. (5) can be used to calculate the "leptonic decay amplitudes" F_n, defined

Table I. Mesons assumed to dominate the spectral functions $\rho_{\alpha\alpha}^{(j)}(\mu^2)$. The masses in the third column are experimental; see Ref. 9. The masses in the last column are predicted: $A1$ in Ref. 1, ω by Eq. (6), and K^* and E by Eq. (14).

Current	Dom. meson $j=0$	Dom. meson $j=1$	Predicted mass for $j=1$
V:$I=1$, $Y=0$	···	$\rho(760)$	input
V:$I=0$, $Y=0$	···	$\omega(780)$	760 MeV
V:$I=\frac{1}{2}$, $Y=\pm1$	κ	$K^*(890)$	825
A:$I=1$, $Y=0$	π	$A1(1080)$	1075
A:$I=0$, $Y=0$	η and η'	$E(1420)$	1440
A:$I=\frac{1}{2}$, $Y=\pm1$	K	$K_A(1320)$	input, gives $F_K/F_\pi=1.2$

by[10]

$$\langle 0|J_\alpha^{\ \mu}(0)|n,p\rangle = F_n p^\mu (2\pi)^{-3/2}(2\omega_n)^{-1/2}. \quad (7)$$

We shall use an estimate[11] of S and Z based on ρ dominance and Eq. (6):

$$S \simeq 2F_\pi^{\ 2},\ Z \simeq 2F_\pi^{\ 2} m_\rho^{\ 2}. \quad (8)$$

Using Eq. (7) and (8) in Eq. (5), we find the old result that $m_{A1}/m_\rho = \sqrt{2}$ and the new results that

$$F_K/F_\pi = [2(1-m_\rho^{\ 2}/m_{KA}^{\ 2})]^{1/2} = 1.16, \quad (9)$$

$$F_\kappa/F_\pi = [2(1-m_\rho^{\ 2}/m_{K*}^{\ 2})]^{1/2} = 0.73, \quad (10)$$

$$[F_\eta^{\ 2}+F_{\eta'}^{\ 2}]^{1/2}/F_\pi = [2(1-m_\rho^{\ 2}/m_E^{\ 2})]^{1/2} = 1.20. \quad (11)$$

Our prediction of F_K/F_π is in good agreement with the experimental value[12] $F_K/F_\pi = 1.28$. Unfortunately, it is not possible to measure F_κ, F_η, or $F_{\eta'}$ in leptonic decays, but our predictions (10) and (11) are of interest, since they can be used together with Goldberger-Treiman relations to determine the various meson coupling constants. For instance, from Eq. (10) we predict that

$$\left|\frac{G_{\kappa N\Lambda}}{G_{\pi NN}}\right| = \left(\frac{m_\Lambda - m_N}{2m_N}\right)\left(\frac{g_V}{g_A}\right)\left(\frac{F_\pi}{F_\kappa}\right) \simeq 0.11 \quad (12)$$

for the coupling of the scalar kaon.

To calculate the masses of the vector and axial-vector mesons, let us go back to the case of a general Lie algebra, and consider a theory of gauge fields $\Phi_{\alpha\mu}$ interacting minimally with spin-zero fields φ_n. The Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}F_{\alpha\mu\nu}F_\alpha^{\ \mu\nu} - \tfrac{1}{2}m_0^{\ 2}\Phi_{\alpha\mu}\Phi_\alpha^{\ \mu} - \tfrac{1}{2}(\partial_\mu\varphi - g_0 T_\alpha \Phi_{\alpha\mu}\varphi)_n(\partial^\mu\varphi - g_0 T_\beta \Phi_\beta^{\ \mu}\varphi)_n + \mathfrak{F}(\varphi), \quad (13)$$

where $F_{\alpha\mu\nu}$ is the usual antisymmetric gauge-invariant derivative of $\Phi_{\alpha\mu}$, m_0 is the bare mass, and T_α is the matrix representing the Lie algebra on the φ_i. In models of broken symmetry like the σ model,[13] the vacuum expectation value of φ_n is a nonzero vector of order minus one in coupling constants; so (13) shows that to zeroth order the mass matrix of $\Phi_{\alpha\mu}$ will be

$$m_{\alpha\beta}^{\ \ 2} = m_0^{\ 2}\delta_{\alpha\beta} + g_0^{\ 2}(T_\alpha\langle\varphi\rangle_0)_n(T_\beta\langle\varphi\rangle_0)_n. \quad (14)$$

We have almost succeeded in proving that Eq. (14) is actually valid (within the meson dominance approximation) to all orders in coupling constants.

The natural extension[14] of the σ model to SU(3) $\otimes$ SU(3) puts the scalar and pseudoscalar mesons in an 18-dimensional representation $(\underline{3},\underline{3}^*) + (\underline{3}^*,\underline{3})$. In this case there are just two fields that can have nonvanishing vacuum expectation values, a scalar singlet σ_1 and the $I=Y=0$ component σ_8 of the scalar octet. Setting $\langle\sigma_8\rangle_0 = 0$ yields the results of exact SU(3), while keeping terms of first order in $\langle\sigma_8\rangle_0$ reproduces the Gell-Mann–Okubo relation. In fact, $\langle\sigma_8\rangle_0 \simeq -0.3\langle\sigma_1\rangle_0$, and we are keeping all order in SU(3) symmetry breaking. With the experimental values[9] of m_ρ and m_{KA}, and our previous prediction that $m_{A1}/m_\rho = \sqrt{2}$, we predict the values given in Table I.[15] We do not find the D, just as before we did not find the φ, but aside from this Eq. (14) appears to furnish a successful account of the observed vector and axial-vector mesons.

We are grateful for discussion with S. Coleman, K. Johnson, F. E. Low, and J. Schwinger. One of us (S.W.) would like to thank the Physics Department of Harvard University for their hospitality.

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[1]S. Weinberg, Phys. Rev. Letters 18, 507 (1967).

[2]J. Das, G. S. Guralnik, V. S. Mathur, F. E. Low, and J. E. Young, Phys. Rev. Letters 18, 759 (1967).

[3]We will be particularly concerned with chiral SU(2) $\otimes$ SU(2) and SU(3)$\otimes$ SU(3). These are semisimple algebras, but parity conservation allows our proofs to go through as if the algebras were simple. We are using a canonical basis, in which $C_{\alpha\beta\gamma}$ is totally antisymmetric and the invariant tensor is $\delta_{\alpha\beta}$.

[4]J. Schwinger, Phys. Rev. Letters 3, 296 (1959).

[5]T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters 18, 1029 (1967).

[6]This is not always justified; see K. Johnson and F. E. Low, Progr. Theoret. Phys. Suppl. 37-38, 74 (1966). However, there is no particular reason to doubt the propriety of using the Jacobi identity here, since it does not lead to contradictions, but in fact gives results which can also be obtained by other meth-

ods (such as perturbation theory) which do not employ the Jacobi identity.

[7]K. Johnson, Nucl. Phys. 25, 431 (1961); S. Okubo, Nuovo Cimento 44B, 1015 (1966); D. G. Boulware and S. Deser, Phys. Rev. 151, 1278 (1966).

[8]For a different approach, see T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters 18, 761 (1967).

[9]The meson masses are taken from the latest compilation of A. H. Rosenfeld et al., Rev. Mod. Phys. 39, 1 (1967). The K_A, $A1$, and E are not definitely known to be 1^+ resonances, but in each case this is a plausible interpretation of the data. [There no longer seems to be any problem in understanding the $A1$ width; J. Schwinger, Phys. Letters 24B, 473 (1967); H. Schnitzer and S. Weinberg, to be published.] If there are several mesons dominating a particular $\rho_{\alpha\alpha}^{(0)}(\mu^2)$, then our predicted value for F_n really gives the rms value of the corresponding F_n's.

[10]Only one of the $J_\alpha{}^\mu$ can couple to any given meson; so it is not necessary to keep a label α on F_n.

[11]K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters 16, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. 147, 1071 (1966); F. J. Gilman and H. J. Schnitzer, Phys. Rev. 150, 1362 (1966); J. J. Sakurai, Phys. Rev. Letters 17, 552 (1966); M. Ademollo, Nuovo Cimento 46A, 156 (1966). This estimate is based on several questionable approximations (for a discussion, see D. A. Geffen, to be published), but it does seem to work well.

[12]For a review of the experimental determination of F_K/F_π, see the rapporteur's talk by N. Cabbibo, in Proceedings of the Thirteenth International Conference on High Energy Physics, Berkeley, California, 1966 (University of California Press, Berkeley, 1967), p. 29. We shall show in a separate article that SU(3) symmetry breaking reduces the $K \to e + \nu + \pi$ decay amplitude by about 10%, so that $\tan\theta$ is about 10% larger than had been thought, and the "experimental" value of F_K/F_π is about 1.15 rather than 1.28.

[13]J. Schwinger, Ann. Phys. (N.Y.) 2, 407 (1957); M. Gell-Mann and M. Levy, Nuovo Cimento 16, 705 (1960).

[14]The predictions of this model for scalar and pseudoscalar masses and coupling constants will be discussed in a separate article by S. Glashow and S. Weinberg, to be published.

[15]If we suppose that the average mass contributing to the $\Delta S = 1$ vector spectral function is 825 MeV rather than 890 MeV, then the prediction (11) becomes $F_K/F_\pi \cong 0.5$, and (13) becomes $|G_{KN\Lambda}/G_{\pi NN}| \simeq 0.15$.

ERRATA

$\bar{p}p$ ELASTIC SCATTERING FOR INCIDENT MOMENTA BETWEEN 1.0 and 2.50 BeV/c. B. Barish, D. Fong, R. Gomez, D. Hartill, J. Pine, and A. V. Tollestrup [Phys. Rev. Letters 17, 720 (1966)].

A mistake on the verticle scale of the graph in Fig. 1 was made. The scale should be multiplied by $\frac{1}{2}$. We are indebted to Ling-Lie Wang for this discovery.

SATURATION OF SUPERCONVERGENCE RELATIONS AND CURRENT-ALGEBRA SUM RULES FOR FORWARD AMPLITUDES. Frederick J. Gilman and Haim Harari, [Phys. Rev. Letters 18, 1150 (1967)].

A square root sign was dropped due to a typographical error in the sentence following Eq. (8), which should read: "where $f_\pi = 135$ MeV is the decay constant of the charged pion, predicted by PCAC to satisfy $f_\pi = (2)^{1/2} G_A m_N g_{\pi N}$."