INVESTIGATION OF THE HYPOTHESES OF KHURI'S THEOREM ON REGGE-POLE ASYMPTOTES*

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The hypotheses of Khuri's theorem, which imply that asymptotes of Regge trajectories cannot be infinite, are studied. We show that the residue function $\beta(s)$ is expected to have an essential singularity at infinity and that $\alpha(s)$ may well have complex branchpoints, making Khuri's theorem consistent with infinite asymptotes.

Recently, there has been considerable interest in the possibility that Regge-trajectory functions $\alpha(s)$ behave asymptotically as

$$\mathrm{Re}\,\alpha(s)\to\pm\infty,\qquad s\to\pm\infty. \tag{1}$$

The behavior (1) differs from that predicted by most models, for example, potential theory, where $\alpha(s)$ approaches the poles of the Born approximation at negative integers as $s\to\pm\infty$. However, trajectories with the properties of (1) are strongly suggested by experiment where a large number of high-energy Regge recurrences have been found, and $\mathrm{Re}\,\alpha$ appears to be roughly linear over the range of observable energies.[1]

Also, several parallel theoretical developments are beginning to confirm (1). Chu and Tan,[2] acting on a suggestion of Chew and Jones, have given a simple bootstrap model for trajectories that satisfy (1). The model is based upon the importance at high energies of two-body channels with high spin and relatively low orbital angular momentum. Mandelstam[3] has made a very plausible argument in potential theory that if the scattering particles are composite, trajectories will have the behavior (1). Also in a theory which possesses a third double-spectral function, the present authors[4] have pointed out that because of the presence of Regge-Mandelstam cuts,[5] $\alpha(s)$ may pass freely through fixed poles of the amplitude located in J at nonsense wrong-signature points.

Khuri has recently made an important contribution to this subject by studying in a more systematic way the consistency conditions imposed by (1) on the analyticity and asymptotic behavior of scattering amplitudes and Regge functions. Khuri shows that the standard assumptions concerning analyticity and asymptotic behavior of $\alpha(s)$ and $\beta(s)$, the Regge-trajectory and residue functions, are in contradiction to (1). In this note we examine in some detail the properties of $\beta(s)$ and $\alpha(s)$ and suggest that the standard assumptions are probably wrong and that the behavior (1) is consequently consistent.

The results of Khuri[6] require that if (1) holds, at least one of the following statements (standard assumptions) be false:

(i) The amplitude $A(s,t)$ is analytic in the cut s plane and is bounded for fixed t by $f(s) = c\exp(|s|^{\frac{1}{2}-\epsilon})$; (ii) $A(s,z)$ is bounded by $f(s)$ for fixed z; (iii) the Sommerfeld-Watson transformation of the partial-wave amplitude $a(l,s)$ exists, and $a(l,s)$ is bounded by $f(s)$ for fixed l; and (iv) $\alpha(s)$ and $\beta(s)$ are analytic with a single cut from $s=4$ to $s=\infty$, $\alpha(s)$ is polynomial bounded, and $\beta(s)$ is bounded by $f(s)$.[7]

We now discuss the possibility that hypothesis (iv) is not fulfilled and that (1) is valid.[8]

First we consider the asymptotic behavior of $\beta(s)$. While the bound $f(s)$ in (iv) appears to to be a generous one, it is, in fact, not plausible if $\alpha(-\infty)=-\infty$. There are two reasons for this: The first is the Mandelstam l-plane symmetry which states that[9]

$$a(l,s)=a(-l-1,s),\qquad l=\tfrac{1}{2},\tfrac{3}{2},\tfrac{5}{2},\cdots. \tag{2}$$

This symmetry was shown by Mandelstam for potential scattering.[9] Its truth in the relativistic case has been freely conjectured. If it is not true, then fixed negative half-integer powers are introduced into the asymptotic behavior of the amplitude which would negate the attractive idea that all asymptotic exponents vary with momentum transfer. [The Mandelstam symmetry can be understood formally in the Froissart-Gribov transformation for $a(l,s)$ from the symmetry of the Q functions, $Q_l=Q_{-l-1}$ $l=\frac{1}{2},\frac{3}{2}\cdots$.]

The Mandelstam symmetry requires that $\beta(s)$ vanish for all negative s where $\alpha(s)$ equals a negative half-odd integer less than $-\frac{1}{2}$ and for which there is no compensating trajectory for the same value of s at $J=-\alpha(s)-1$. Thus, if $\alpha(s)\to-\infty$ as $s\to-\infty$, $\beta(s)$ will have an infinite accumulation of zeros at $s=-\infty$. Such a function necessarily has an essential singularity at infinity. We can establish immediately a lower bound on $\beta(s)$ as $s\to-\infty$ in the left-half plane by the following application of Carlson's theorem:

Consider $\beta(s)$ as a function of $\alpha(s)$. Then $\beta(\alpha)$ is analytic for $\mathrm{Re}\,\alpha$ sufficiently negative and possesses zeros for $\alpha=-\frac{3}{2}, -\frac{5}{2}$, and $\cdots$. If $\beta(\alpha)$ is not identically 0, then by Carlson's theorem,[10]

$$\beta(\alpha)\underset{\alpha\to-\infty}{\geqslant}\mathrm{const}\,e^{\pi|\alpha|}. \tag{4}$$

Hence, in order for β to have the behavior $\beta<f(s)$ and to satisfy the conditions of (iv), α must clearly decrease more slowly than $-\sqrt{-s}$ as $s\to-\infty$. However, Khuri shows[6] that if $\alpha<|\sqrt{s}|$ as $s\to\infty$, then $\mathrm{Re}\,\alpha$ must approach $+\infty$ as $s\to-\infty$ if it approaches $+\infty$ as $s\to+\infty$.

To see roughly what is involved, suppose that

$$\alpha(s)\underset{s\to-\infty}{\longrightarrow}-(-s)^{p}, \tag{5}$$

where p is some real positive power. Then clearly

$$\alpha(s)\underset{s\to+\infty}{\longrightarrow}-(s)^{p}[\cos\pi p-i\sin\pi p]. \tag{6}$$

Thus, unless $p>\frac{1}{2}$, $\mathrm{Re}\,\alpha(s)$ approaches the same limit as $s\to\pm\infty$. Thus if $\mathrm{Re}\,\alpha$ is to behave as (1) and be polynomial bounded with only a right-hand cut, it must approach $\pm\infty$ faster than $\sqrt{s}$ and (iv) is violated.

We proceed to show, by taking a simple example for $\beta(s)$, that none of Khuri's hypotheses other than (iv) are obviously violated[8] if α approaches $+\infty$ faster than $\sqrt{s}$. Consider, as a simple example of a $\beta(s)$ having the required properties,

$$\beta(s)=\mathrm{const}/\Gamma(\alpha(s)-\tfrac{3}{2}). \tag{7}$$

Khuri demonstrates that statements (i)-(iii) imply

$$\lim_{\substack{s\to\infty\\ \text{fixed } z}}|\beta(s)P_{\alpha(s)}(z)|<f(s), \tag{8}$$

In Eq. (4), we are using the fixed-z limit required by (ii) which is harder to satisfy than the fixed-t limit. (See Ref. 7.) Condition (8) is impossible to satisfy if (1) and also (iv) are true. [The essential reason is that if $|\beta(s)|<f(s)$, then a theorem of Boas[11] shows that $|\beta(s)|$ must be bounded below by $\exp(-\epsilon(s)^{1/2}/\ln s)$.] However, if (7) is true, the requirement of (8) can be easily met if $\alpha(s)\to\infty$ as $s\to\infty$, since

$$\beta(s)\underset{s\to\infty}{\longrightarrow}\mathrm{const}\,e^{-\alpha(s)\ln s} \tag{9}$$

and

$$|P_\alpha(z)|\underset{\alpha\to\infty}{\leq}e^{b\alpha},$$

where $b>0$.

We see from (9) that (8) is true even if $\alpha(s)$ approaches $+\infty$ faster than $s^{1/2}$ as $s\to\infty$. [The restrictions of the Boas theorem no longer apply, since $|\beta(s)|>f(s)$.] Thus, a β given by (7) is consistent with the condition (8), which was implied by (i)-(iii).

A second point which can be made in connection with the asymptotic behavior of $\beta(s)$ concerns the possibility of $\beta(s)$ having an infinite number of poles which accumulate at infinity. It has been shown that $a(l,s)$ has fixed poles at nonsense wrong-signature values of l. Such poles depending upon dynamics may be manifested in $\beta(s)$ at values of s such that $\alpha(s)$ is a nonsense wrong-signature value.[11] Such behavior undoubtedly also contributes to the bound in (iv) becoming worse.

These two arguments suggest that in the presence of the behavior (1), the bound (iv) on $\beta(s)$ is invalid and must be replaced by (4). This latter behavior is not in contradiction with any known, more basic, truths. It does, however, raise (but not settle) the question of the existence of partial-wave dispersion relations for $a(l,s)$ or, at least, the uniformity in l of its polynomial boundedness in s.[12] It also invites comparison with the $f(s)$ bound on form factors proven by Jaffe.[13] However, the general relation between form factors and Regge residue functions is unknown, while in weak processes, form factors have shown to be related, not directly to Regge residues but to associated fixed J-plane pole residues.[14]

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[1]A. Citron et al., Phys. Rev. 144, 1101 (1966); S. W. Kormanyos et al., Phys. Rev. Letters 26, 709 (1966).

[2]S. Chu and C. Tan, University of California Radiation Laboratory Report No. UCRL-17511, 1967 (unpublished).

[3]S. Mandelstam, University of California Radiation Report No. UCRL-17250, 1966 (unpublished).

[4]C. E. Jones and V. L. Teplitz, Phys. Rev. (to be published).

[5]S. Mandelstam, Nuovo Cimento 30, 1148 (1963).

[6]N. N. Khuri, "On the Possibility of an Infinite Sequence of Regge Recurrences" (to be published).

[7]In Ref. 6, it was assumed that $\gamma(s)=\beta(s)(s-4)^{-\alpha(s)}$ was bounded by $f(s)$. We are indebted to Professor Khuri for pointing out to us that his argument may be equally well carried out by assuming that $\beta(s)$ is bounded by $f(s)$. The additional convergence which appears to be present in the latter case because of the absence of the factor $(s-4)^{\alpha}$ is negated by using the fixed-z limit for $P_\alpha(z)$ as we have done in Eq. (8), rather than the fixed-t limit of Ref. 6.

[8]Hypotheses (iv) and (iii) appear superficially not to be independent. If, for example, $\beta(s)>f(s)$, unless there are special cancellations, $a(l,s)$ will also exceed $f(s)$, in which case normal partial-wave dispersion relations become invalid. More precise statements are difficult to make, and we will treat (iii) and (iv) as independent below. We are grateful to Professor Khuri for discussions on this point.

[9]S. Mandelstam, Ann. Phys. (N.Y.) 19, 254 (1962).

[10]See E. C. Titchmarsh, Theory of Functions (Oxford University Press, New York, 1939), Sec. 5.81.

[11]R. P. Boas, Jr., Entire Functions (Academic Press, Inc., New York, 1954).

[12]Ref. 4, and also, S. Mandelstam and L. Wang, Phys. Rev. (to be published).

[13]A. M. Jaffe, Phys. Rev. Letters 17, 661 (1966).

[14]J. B. Bronzan, I. S. Gerstein, B. W. Lee, and F. E. Low, Phys. Rev. Letters 18, 32 (1967); and V. Singh, ibid. 18, 36 (1967).

FORWARD PEAKS FROM THE EXCHANGE OF ODD-PARITY REGGE TRAJECTORIES*

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It is shown that a parity-doubled (at $t=0$), $I=1$, odd-signature trajectory is favored by a simple dynamical mechanism. A set of experiments is described whereby one could settle decisively the question of which (if any) of the possible O(4) conspiracies at $t=0$ is realized.

(1) There has been considerable interest recently in the possibility that the exchange of a pion Regge trajectory could give rise to a forward peak in reactions such as $n+p\rightarrow p+n$, $\gamma+N\rightarrow\pi+N$.[1-4] Such a contribution of an odd-parity trajectory to forward processes must necessarily involve conspiracy between trajectories degenerate at $t=0$, of the kind predicted by O(4) symmetry.[5-8] If one of the leading conspirators is to be the pion trajectory, the conspiracy must be of class III, in the terminology of Ref. 8. This is the parity-doubled class, in which the pion $[G=P=(-1)^{J+1}]$ conspires with a trajectory for which $G=-P=(-1)^{J+1}$ (at the same leading point, $J=\alpha$).

However there is a grave theoretical difficulty with this idea, as can be seen from the simplest O(4) consideration. The two angular momenta (j_1,j_2) which characterize class III differ by unity. ($j_1+j_2=p$, the index in which the amplitude is to be analytically continued; $|j_1-j_2|=M$, which is to remain fixed.[7,8]) For $M=1$, the minimum physical angular momentum in a representation is unity. Thus, if the pion trajectory were bound with $J=0$ at zero mass and were in class III, we would necessarily conclude that the pion was unobservable. Of course, away from zero energy the T matrix will no longer be diagonal in the O(4) labels, but the pion mass is so small that the class-III trajectory should be nonsense choosing at $J=0$.

(2) It may nevertheless be of great interest to look at the possibility that a class-III, $I=1$ conspiracy does exist, with intercept $\alpha(0)\approx 0$,[9] but giving rise to no $J=0$ particles. We would then speculate about possible $J=1$ or $J=2$ intersections, depending on the signature. Note that which of the possible meson pairs can couple to the trajectories depends on the signature. Remembering that the distinctive feature of class-III exchange in forward scattering is the fact that the s^α contribution is pure spin double flip,[10] we see that observing which (if either) of the forward reactions $\pi+N\rightarrow\rho+N$ or $\pi+N\rightarrow\omega+N$ shows large spin double flip would indicate the signature of the trajectory.[11]