123 (1966).

[6]M. Flicker and E. H. Lieb, Phys. Rev. 161, 179 (1967).

[7]E. Lieb, Phys. Rev. 130, 1616 (1963).

[8]C. N. Yang and C. P. Yang, to be published. See also C. N. Yang, in Proceedings of Eastern Theoretical Conference, November, 1966, Brown University (W. A. Benjamin, Inc., New York, 1967), p. 215.

EXACT DYNAMICS OF LANDAU ELECTRONS

Herbert F. Budd
Laboratoire de Physique des Solides, Ecole Normale Supérieure, Paris, France*
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The availability of intense monochromatic radiation sources has stimulated both experimental and theoretical studies of strong-field effects in solids. The theoretical problem is that of calculating the dynamics of charge carriers in strong electric fields, for which simple perturbation theory is inadequate.

We treat exactly the dynamics of electrons in crossed electric and magnetic fields, the electric field being spatially uniform but otherwise arbitrary. We consider only the case of spherical constant-energy surfaces and neglect interband effects, thus limiting our analysis to intraband dynamics.

A recent paper by Hanamura, Lax, and Shin[1] has dealt with this problem for the special case of a sinusoidal electric field; however, their results are unfortunately incorrect. The exact solutions given here are readily understandable and are essentially identical to the results one would obtain classically.

We take the magnetic field along the z direction, the electric field along the x direction, and we work in the Landau gauge $\vec{A}=[0, +Hx, 0]$. The Hamiltonian in the presence of a spatially uniform electric field $E(t)$ is simply

$$\mathcal{H}=(P_x^{\,2}+P_z^{\,2})/2m+\tfrac{1}{2}m\omega_c^{\,2}(x-x_0)^2-eE(t)x, \qquad (1)$$

where $\omega_c=eH/mc$ and $x_0=-P_y/m\omega_c$ is the orbit center. We wish to calculate the evolution operator U for the Hamiltonian (1), where

$$|t\rangle=U|i\rangle;\quad dU/dt=-\beta\mathcal{H}U;\quad \beta=i/\hbar. \qquad (2)$$

$|t\rangle$ is the state at the time $t\geq 0$, given that the initial state $(t=0)$ is $|i\rangle$. Since P_z and P_y commute with $\mathcal{H}$, they are constants of motion and will be taken equal to zero for convenience. This makes $x=0$, and we then have the simple one-dimensional harmonic-oscillator Hamiltonian with a driving force,·

$$\mathcal{H}=P_x^{\,2}/2m+\tfrac{1}{2}m\omega_c^{\,2}x^2-eE(t)x. \qquad (3)$$

There are various methods available for obtaining the exact evolution operator corresponding to this Hamiltonian. Louisell[2] has presented a solution using normal ordering techniques, and similar results are obtainable from the generalized Baker-Hausdorff[3] formula. We present here a simple and physical method for obtaining U. We take U in the following form:

$$U=e^{-\beta D}e^{\beta\alpha x}e^{-\beta\gamma P_x}e^{-\beta H_0 t}, \qquad (4)$$

where α, γ, and D are c numbers depending on time and H_0 is the Hamiltonian of Eq. (3) with $E(t)=0$. Evaluating dU/dt and requiring that Eq. (2) be satisfied, we obtain

$$\frac{d^2\gamma}{dt^2}+\omega_c^{\,2}\gamma=\frac{eE(t)}{m};\quad \alpha=m\frac{d\gamma}{dt};$$

$$\frac{dD}{dt}=\frac{\alpha^2}{2m}-\frac{m\omega_c^{\,2}}{2}\gamma^2. \qquad (5)$$

Here $\gamma(t)$ satisfies the classical driven-harmonic-oscillator equation and plays the role of the classical particle displacement, while $\alpha(t)$ plays the role of the classical momentum. It is immediately obvious that a sinusoidal driving field can only result in a time dependence of α and γ which has frequency components at the driving frequency and at the resonant frequency. There are no sum and difference frequencies as has been reported in Ref. 1.

The requirement that $U(0)=1$ is readily met by choosing $\alpha(0)=\gamma(0)=D(0)=0$. The corresponding solutions of Eq. (5) are

$$\gamma = \frac{e}{m\omega_c}\int_0^t E(s)\sin\omega_c(t-s)ds,$$

$$\alpha = e\int_0^t E(s)\cos\omega_c(t-s)ds;$$

$$D = \int_0^t \left[\frac{\alpha^2(s)}{2m} - \frac{m\omega_c^{\,2}}{2}\gamma^2(s)\right]ds. \qquad (6)$$

Let us now examine the evolution of an initial state $|n\rangle$ which is an eigenstate of H_0, i.e., $H_0|n\rangle = E_n|n\rangle = (n+\frac{1}{2})\hbar\omega_c|n\rangle$. The x representatives of these states $\varphi_n(x)$ are well-known Hermite-function solutions of the harmonic oscillator. The x representative of the state at time t is simply

$$\langle x|t\rangle = \langle x|U|n\rangle = e^{-\beta D}e^{\beta\alpha x}e^{-\beta E_n t}\varphi_n(x-\gamma). \qquad (7)$$

Thus the center of the cyclotron orbit moves just as does the classical particle, $|\langle x|t\rangle|^2 = \varphi_n^{\,2}(x-\gamma)$, and the expectation value of the x component of the velocity is again identical to the classical result:

$$\langle t|V_x|t\rangle = -\frac{i\hbar}{m}\int dx\,\varphi_n(x-\gamma) \times\left[\frac{i\alpha}{\hbar}\varphi_n(x-\gamma) + \frac{d}{dx}\varphi_n(x-\gamma)\right] = \frac{\alpha}{m}. \qquad (8)$$

The results for a sinusoidal driving force are readily calculated, and of particular interest is the case of a resonant excitation $E(t) = E_0 \times \sin\omega_c t$. The power absorption in this case is simply

$$P = \frac{e^2E_0^{\,2}t}{2m}[\sin^2\omega_c t], \qquad (9)$$

which diverges for large t, just as for an undamped classical oscillator. We also readily verify that our evolution operator [Eq. (4)] for the case of an adiabatically applied constant electric field

$$E(t) = \lim_{n\to 0^+} E_0(1-e^{-nt})$$

generates the eigenstates of the steady-crossed-field problem.[4]

For the sake of completeness we include the transition probabilities from the ground-state Landau level to any excited state:

$$C_n(t) = |\langle n|U|0\rangle|^2 = \frac{[\epsilon(t)/\hbar\omega_c]^n}{\hbar!}\exp[-\epsilon(t)/\hbar\omega_c],$$

where

$$\epsilon(t) = |\int_0^t eE(s)\exp(i\omega_c s)ds|^2/2m = \int_0^t P(t')dt'.$$

Here $\epsilon(t)$ is simply the classical energy of the oscillator at time t. We note that at the resonant frequency $\epsilon(t)$ diverges for large t [Eq. (9)] and $C_n \to 0$ for finite n. This simply corresponds to the continual excitation of the oscillator to higher and higher quantum states.

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*Associated with Centre National de la Recherche Scientifique.

[1]E. Hanamura, B. Lax, and E. E. H. Shin. Phys. Rev. Letters 17, 923 (1966).

[2]W. Louisell, Radiation and Noise in Quantum Electronics (McGraw-Hill Book Company, Inc., New York, 1964), p. 123.

[3]G. H. Weiss and A. A. Maradudin. J. Math. Phys. 3, 77 (1962).

[4]A. H. Kahn and H. R. R. Frederikse, Solid State Phys. 9, 270 (1959).