# BOOTSTRAP OF THE $\rho$ REGGE TRAJECTORY

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The amplitude for $\omega+\pi\rightarrow\pi+\pi$ is considered within a dynamical scheme proposed by Mandelstam, based on rising Regge trajectories, the narrow-resonance approximation, and generalized superconvergence relations. The $\rho$ trajectory is shown to qualitatively bootstrap itself. Also, a world consisting only of the particles on a vacuum trajectory is shown to be inconsistent within this approximation.

Mandelstam has recently proposed a dynamical scheme[1] based on approximating the amplitude by a finite number of Regge poles in all channels. Crossing is imposed by generalized superconvergence relations.[2,3] In the first approximation, which is essentially the narrow-resonance approximation (NRA), the trajectories are assumed to be straight lines and unitarity determines the Regge residue up to an entire function, which can then be approximated by a finite polynomial. This approximation, which can be systematically improved, allows one to derive algebraic relations between a finite number of parameters. The relations may be sufficient to determine these parameters self-consistently.

We apply the above scheme to the amplitude $\omega+\pi\rightarrow\pi+\pi$, where only one trajectory (i.e., that with $I^{GC}=1^{+-}$, $P=-1$, and negative signature, the quantum numbers of the $\rho$ trajectory) can contribute in all channels.[4] It is shown that this trajectory can indeed bootstrap itself, and that one obtains reasonable values for the Regge parameters. Within the same approximation, we show that a universe consisting only of the particles on a vacuum trajectory cannot be self-consistent.

I. Bootstrap of the $\rho$ trajectory.—In the reaction $\omega+\pi\rightarrow\pi+\pi$, there is one independent amplitude which we can take to be the $t$-channel helicity amplitude $f_{0\lambda,00}{}^t(s,t)$, $\lambda=\pm1$. We have

$$\begin{aligned} f_{0\lambda,00}{}^{t}(s,t) &= f_{0-\lambda,00}{}^{t}(s,t) \\ &= \sum_{J=1}(2J+1)G_{0\lambda,00}{}^{J}(t)d_{0-\lambda}(Z_t) \\ &= f_{00,\lambda0}{}^{s}(s,t). \end{aligned} \tag{1}$$

We define $\tilde{f}(st)$ to be the amplitude which is free of kinematical singularities in $s$ and $u$ for fixed $t$,[5]

$$\tilde{f}(s,t,u)=\csc\theta_t f_{0\lambda,00}{}^{t}(s,t)=\tilde{f}(u,t,s). \tag{2}$$

The basic assumption is[1] that $G^J(t)$ can be approximated by one "$\rho$" trajectory

$$G_{0\lambda,00}{}^{J}(t)=\beta(t)/[J-\alpha(t)], \tag{3}$$

and that $\alpha(t)=at+b$. Unitarity and analyticity in the NRA then determine the form of $\beta(t)$ up to an entire function $E(t)$,

$$\beta(t)=\left(\frac{4q_tq_{t'}a}{e}\right)^{at+b}\frac{t^{\frac{1}{2}}[(at+b)(at+b+1)]^{\frac{1}{2}}}{\Gamma(at+b+\frac{3}{2})}E(t). \tag{4}$$

This form exhibits the threshold behavior of $\beta(t)$, the nonsense-eliminating factors[6] at $\alpha(t)=0$, $-1$, and at negative half-integers, and a scale factor $e/4a$ ($e=2.718$) which makes the left-hand side of (4) behave like $t^{1/2}E(t)$ for large $t$. The additional factor of $\sqrt{t}$ is due to the complication of unequal mass in the $\pi\omega$ channel together with helicity flip.[7]

We then have that

$$\operatorname{Im}\tilde{f}(s,t)\underset{s\to\infty}{\to}\frac{2\pi^{\frac{1}{2}}q_tq_{t'}\sqrt{t}}{s\Gamma(\alpha(t))}\left(\frac{4as}{e}\right)^{\alpha(t)}E(t). \tag{5}$$

We now impose crossing by means of the simplest finite-energy, generalized superconvergence relation for $\tilde{f}(s,t)$, which is[3]

$$\int^N ds\,(s-u)\operatorname{Im}\tilde{f}(s,t) = \frac{\pi^{\frac{1}{2}}NE(t)}{[\alpha(t)+1]\Gamma(\alpha(t))}(4q_tq_{t'}\sqrt{t})\left(\frac{4aN}{e}\right)^{\alpha(t)}, \tag{6}$$

where $N$ is a suitable cutoff, to be discussed below. Saturating the left-hand side by the first particle on the "$\rho$" trajectory, at $J=1$, we derive the rather simple relation ($\Sigma=3m_\pi^2$

$+m_\omega^2 \approx 0.67\ \mathrm{BeV}^2$)

$$\frac{E(t)}{E[(1-b)/a]}=\frac{4(at+b+1)\Gamma(at+b)[t+2(1-b)/a-\Sigma]}{Ne}\left(\frac{e}{4Na}\right)^{at+b}. \tag{7}$$

Even without saying anything more about $E(t)$, evaluation of this equation at $t=-b/a$, $\alpha(t)=0$, and at $t=(1-b)/a$, $\alpha(t)=1$ yields

$$(2-3b)/a=\Sigma; \quad Na=\sqrt{2}. \tag{8}$$

In this form of a self-consistent bootstrap, $N$ is not arbitrary; it must surely lie above $(1-b)/a$ and below $(3-b)/a$. Since we are neglecting daughter trajectories (clearly important here near $t\approx 0$) other trajectories with identical quantum numbers, $N$ should perhaps be taken lower than $(2-b)/a$ (where the first resonance would lie on a parallel daughter trajectory). We therefore choose $N$ in the interval

$$(3-2b)/2a \leq N \leq (4-2b)/2a. \tag{9}$$

For all values of $N$ in this interval, (8) has a consistent solution with $b\leq 1$ and $a>0$. Indeed, as $N$ increases in this interval, $b$ increases from 0.1 to 0.6, $a$ decreases from 2.6 to 0.34 $\mathrm{BeV}^{-2}$, and $m_\rho^2=(1-b)/a$ increases from 0.35 to 1.34 $\mathrm{BeV}^2$. It is also satisfying to note that $E(t)$ is almost exactly a constant in the interval $-b/a\leq t\leq(1-b)/a$ and depends only on the trajectory $\alpha(t)$ independent of the value of $N$! We have

$$\frac{E(t)}{E((1-b)/a)}=\frac{\Gamma(\alpha(t)+2)}{2}\left(\frac{e}{4\sqrt{2}}\right)^{\alpha(t)-1},$$

and $E(t)$ changes by only 4% as $\alpha(t)$ varies from 0 to 1. Note that in this lowest approximation we cannot say anything about the absolute value of $E$ and, like all other bootstrap models, have no way of knowing whether the $\rho$ trajectory will continue to be self-consistent when other channels and particles are added.

It is also simple to show that this bootstrap is stable under spontaneous breaking of isospin invariance for the $\rho$ trajectory and residues. If we only assume charge conservation and $C$ invariance for the trajectories and residues, then in the above approximation we derive three equations of the form (7) which, in the approximation of constant $E$'s, guarantee isospin invariance independent of $N$.

II. Bootstrap of vacuum trajectory.—The simplest of all possible worlds consistent with the above dynamics is described by one vacuum trajectory $\alpha(t)=ct+d$. Any bootstrap dynamics must explain why the real world is more complicated.[8] If we look at the scattering of the lowest members of such a trajectory, i.e., scalars with mass $m^2=-d/c$, in the same approximation as before, we derive

$$\frac{F(t)}{F(m^2)}=\frac{[c(t-m^2)+2][c(t-m^2)+1](t-2m^2)}{N^2c}\left(\frac{e}{4cN}\right)^{c(t-m^2)}, \tag{10}$$

where the residue is now

$$\beta(t)=\left(\frac{4q_t^2c}{e}\right)^{ct+d}\frac{F(t)}{\Gamma(ct+d+\frac{3}{2})}.$$

Evaluation of (10) at $t=m^2$ yields

$$1=-2m^2/N^2c,$$

which is inconsistent with a positive slope, independent of the choice of $N$!

III. Conclusions.—We have shown that application of Mandelstam's scheme to the simplest cases of a complete bootstrap of a trajectory yields qualitatively satisfying results, independent of the exact choice of the cutoff. However, quantitatively the lowest approximation is extremely cutoff dependent. It is not at all clear that the inclusion of more channels and particles will make self-consistent quantitative results less cutoff dependent.

It is a pleasure to thank Dr. M. Ademollo for valuable discussions.

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[1]S. Mandelstam, to be published.

[2]K. Igi, Phys. Rev. Letters 9, 76 (1962).

[3]D. Horn and C. Schmid, California Institute of Technology Report No. CALT-68-127 (unpublished).

[4]M. Ademollo, H. R. Rubinstein, G. Veneziano, and

M. A. Virasora (to be published) have considered this amplitude and saturated a finite-energy superconvergence relation with the $\rho$ trajectory. Their approach, however, is not one of a systematic bootstrap.

[5]L. L. Wang, Phys. Rev. 142, 1187 (1966); Y. Hara, Phys. Rev. 136, B507 (1966); T. L. Trueman, Phys. Rev. Letters 17, 1198 (1966).

[6]In this lowest approximation, we neglect all complications due to the third double-spectral function; i.e., cuts and fixed poles at nonsense, wrong-signature values of $J$.

[7]When one has unequal masses in one channel, and maximal helicity flip $n$, the partial-wave amplitude $G^{J\pm}(t)$, of definite signature $\pm$, behaves near $t=0$ like

$$G^{J\pm}(t)\approx t\{-\tfrac{1}{2}[\alpha^{\pm}(0)-n]\},$$

where $\alpha^{\pm}$ is the leading trajectory in the $s$ or $u$ channels. To prove this, one assumes the existence of daughter trajectories, so that the amplitude free of $s$ kinematical singularities behaves as $s^{\alpha(0)-n}$ when $t=0$, and derives at $t=0$ behavior of $G^{J\pm}(t)$ from the Froissart-Gribov definition of $G^{J}(t)$, as in D. Z. Freedman and J. M. Wang, Phys. Rev. 153, 1596 (1967).

[8]The impossibility of a scalar bootstrap has been argued within the framework of $N/D$ equations by P. D. B. Collins, Phys. Rev. 136, B710 (1964), and 139, B696 (1965).

## ERRATUM

**FORWARD COMPTON SCATTERING AT HIGH ENERGY AND THE DRELL-HEARN-GERASIMOV SUM RULE.** Norman Dombey [Phys. Rev. Letters 19, 985 (1967)].

Drell[1] has recently pointed out that in linearized theories [assumption (i)] unitarity does not rule out extra real polymonials in the dispersion relations for $f_1(\omega)$ and $f_2(\omega)$. For example, a term $c\omega^2$ may be added to the right-hand side of Eq. (10), where $c$ is a real constant. Terms such as these would invalidate Eq. (8) and drastically change the energy dependence of $(d\sigma/d\Omega)_{0°}$ as $\omega\to\infty$. If the observed energy dependence of forward Compton scattering at high energies (or low energies[1]) indicates the presence of these terms, it will be necessary to carry out the alternative experiment using linearly polarized photons and a polarized proton target outlined in the Letter in order to separate $|f_1(\omega)|$ from $|f_2(\omega)|$.

I would like to thank Professor Drell for emphasizing this point.

[1]S. D. Drell, Introductory address to the International Symposium on Electron and Photon Interactions at High Energies, Stanford, California, 1967 (unpublished).