$\bar{S}^2 \sim 1/N\alpha'$ is the squared average spacing for the unperturbed ensemble.

Thus, the major effect of the presence of a small time-reversal–noninvariant term on the spacing distribution is a change in shape near the origin. Since this is a region of minor probability, and hence large statistical error, it would appear that observation of such a term using statistical methods is very difficult. However, if the experimental measurements reveal that the distribution near the origin is linear for $S > S_0$, where S_0 is the smallest spacing with good statistics, one can conclude that the noninvariant term, if it exists, is at least $S_0/\bar{S}$ times smaller than the invariant term.

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[1]See, for example, C. E. Porter, Statistical Theories of Spectra: Fluctuations (Academic Press, Inc., New York, 1965).

[2]See, for example, C. E. Porter and N. Rosenzweig, Ann. Acad. Sci. Fennicae: Ser. A VI No. 44 (1960).

[3]Current interest in this question was communicated to the authors by M. L. Mehta.

[4]Porter and Rosenzweig, Ref. 2.

[5]J. F. McDonald, thesis, Wayne State University, 1967 (unpublished). This method is also contained in a paper by J. F. McDonald and L. D. Favro, to be published.

[6]Here we treat G as though it were a member of a unitary ensemble. Thus, there are $N(N-1)$ parameters φ_i. The delta functions $\delta(G')$ insure that G is real (orthogonal) and $\frac{1}{2}N(N-1)$ of the φ_i integrations are trivial.

CALCULATION OF DAUGHTER TRAJECTORIES*

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Explicit calculations show that daughter trajectories are very model dependent. It is therefore necessary to be cautious in applying Lorentz symmetry when the energy is not zero.

At zero energy, the Regge poles obtained from the manifestly covariant Bethe-Salpeter equation occur in families; for each "mother" trajectory with a given $l(0)$, there is a sequence of "daughters" with $l_n(0) = l(0)-n$.[1-4] At $s=0$ (s is the squared energy) the odd daughters have residues whose sign is "wrong," meaning that if the trajectories rise through physical values without this sign changing, the associated particle would be a "ghost." This sign problem is implicit in the work of Freedman and Wang[1] and had been emphasized even earlier by Nakanishi[5] (see also Ciaffaloni and Menotti[6]). An additional symmetry occurring for scalar particles of equal mass leads to several special features; in particular, the odd daughters are actually uncoupled from the scattering amplitude. It is to be expected, for reasons discussed below, that the Regge trajectories may, in the general case, behave quite differently from those previously reported for scalar particles[7] with $M_1 = M_2$ or in the nonrelativistic theory,[8] and we undertook some explicit calculations with two unequal scalar particles to demonstrate this.

We write the Bethe-Salpeter equation for an amplitude of angular mementum l in the symbolic form

$$B(l,s)\psi^l(R,\alpha) = 0, \tag{1}$$

where $B(l,s)$ is a fourth-order partial differential operator in R and α depending parametrically on l and s, R is the four-dimensional relative distance, and $t = R\cos\alpha$ is the continued[9] relative time. We assume that the interaction is a superposition of Yukawa potentials. Equation (1) is self-adjoint with the prescription

$$\psi^\dagger(R,\alpha) = \psi(R,\pi-\alpha)^*, \tag{2}$$

where the reflection $t \to -t$ makes the norm indefinite in sign and thereby leads to the possibility of ghost solutions[5] as well as to unfamiliar level-crossing behavior. We represent ψ^l as a superposition of terms of the form

$$\psi_{nk}{}^l(R,\alpha) = C_k{}^{l+1}(\cos\alpha)R^{n+l+k}e^{-\beta R}e^{\gamma R\cos\alpha}\sin^l\alpha, \tag{3}$$

where the C_k^ν are Gegenbauer polynomials. The parameter $\beta=\beta(s)$ was chosen to give a reasonably good fit for $\cos\alpha=0$ to the known amplitudes for scalar photon exchange,[5,9,10] and $\gamma=\gamma(s)$ was chosen to incorporate the differences in the asymptotic behavior of the solution when $\cos\alpha=\pm1$. An expansion in the functions (3) restricts us to the region below the elastic threshold, which we took at $s=4$.

The overlap integrals

$$\psi_{n'k'}^{l\dagger}\psi_{nk}^{l} = \int_0^\pi\int_0^\infty \psi_{n'k'}^{l\dagger}(R,\alpha)\psi_{nk}^{l}(R,\alpha)R^3 dR \sin^2\alpha d\alpha, \quad (4)$$

as well as matrix elements of the Klein-Gordon operators, can be evaluated analytically, and after removal of trivial gamma-function factors are simple functions of l, n', k', n, and k. We used potentials having the form

$$V(R)=V_0[e^{-m_1R}-\lambda e^{-m_2R}]/R^2, \quad (5)$$

rather than the four-dimensional Yukawa potential $Y(R,m)=mV_0K_1(mR)/R$, because the matrix elements of (5) are also simple functions.[11] The exponential potential can be written as

$$V_0\exp(-mR)/R^2 = \int_m^\infty m\pi^{-1}\mu^{-2}(m^2-\mu^2)^{-\frac{1}{2}}\gamma(R,\mu)d\mu^2. \quad (6)$$

From the variantional principle, we obtain

$$\det b(l,s)=0 \quad (7)$$

as the equation for Regge trajectories $l(s)$, where $b(l,s)$ is the matrix of the $\psi^\dagger B(l,s)\psi$ in our basis. We included terms in the amplitude with $0\le n+k\le N$, and examined the stability as N was increased. The leading trajectory was given quite well for $s\lesssim3$ even with $N=1$ or 2, and stable qualitative features were obtained with $N\sim3$ or 4 for $l\lesssim-1$ and $|s|\lesssim3$. The curves in the Figs. 1 and 2 are based on $N=7$ or 8, as required, i.e., 36 or 45 terms, respectively, and for a given s, l is correct to at least ±0.01, except possibly at some turning points and for $|s|\lesssim3$ or $l\lesssim-1$.

In Fig. 1 we show some trajectories obtained with a potential in which λ and m_2 were chosen to approximate the modified Bessel function. [With $\lambda m_2=m_1$, the $1/R$ term is canceled, and with $\lambda=\frac{1}{2}(5^{1/2}-1)$ the volume integrals of the two potentials have the same ratio as the coefficients of $1/R$.[2] In the equal mass case, $M_1=M_2=1$, we then adjusted the strength by

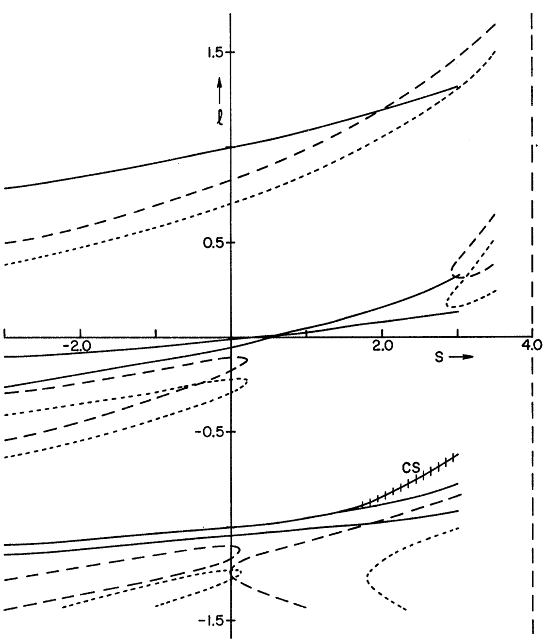

FIG. 1. Regge trajectories for the approximate Yukawa potential. The solid lines pertain to the equal-mass case, for which three of the trajectories were also obtained by CS. The dashed and dotted curves are for $M_2=4M_1$, with $V_0=75$ and $V_0=65$, respectively.

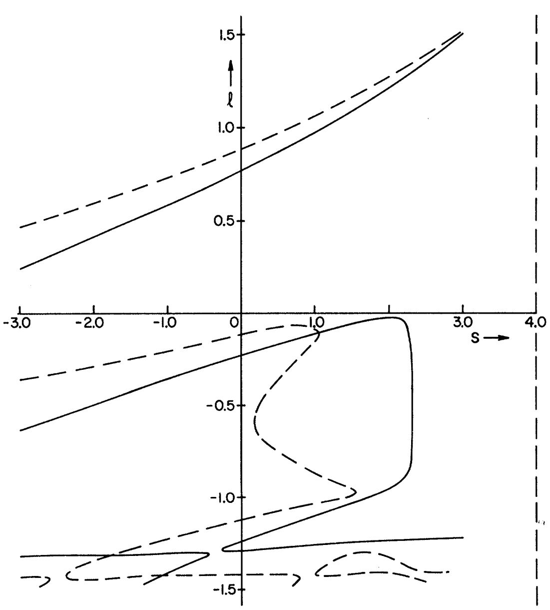

FIG. 2. Trajectories obtained with repulsive-core potentials. The solid curves were obtained with $\lambda=1.82$ and the dashed curves with $\lambda=1.25$, with $V_0=60$ in both cases.

a few percent to obtain $l(0)=1$, as in the calculation of Chung and Snider (CS).[7]] The mother and first daughter agree with CS, although there is a slight discrepancy in the second daughter for $s \gtrsim 1.5$.[12] We also show an "aunt" and its daughter which CS did not compute. Note that the aunt and the first daughter trajectories cross at $s \sim 1$. In the unequal mass case $M_2 = 4M_1$, the breaking of the symmetry mixes the two states; they attract, rather than repel, because the norms of the two states have opposite signs. (The slopes differ because we took $m_1 = M_1$; thus the range of the potential is greater in the unequal mass case.) The behavior of the lower trajectories is similar but very dependent on the potential strength.

The shape of the mixed daughter-aunt trajectories suggests existence of a complex branch connecting the two real branches; this is also indicated by use of degenerate perturbation theory based on the equal-mass solutions. Thus, for $l=0$ the partial wave amplitude evidently has two complex-conjugate poles in s. This is a previously unknown pathology of the ladder approximation; it may be related to the difficulty with unitarity noted by Levine, Wright, and Tjon.[13] (There is, of course, also a pole with $s<0$.)

One expects a repulsive force at short distances[14] to depress the aunt trajectories, but, for reasons of continuity, not to eliminate strange trajectory-mixing phenomena. To check this, we calculated with $m_1 = M_1$, and $m_2 = M_2 = 4M_1$, for $\lambda = 1.25$ and $\lambda = 1.82$ ($V_0 = 60$), and obtained the curves shown in Fig. 2. Note that the first two daughters lie on the same trajectory. There is no additional real branch in the region $2 \lesssim s \lesssim 3.5$, $1 \lesssim l \lesssim -1$; there are, of course, complex branches, but they might not occur on the physical sheet when $l=0$.

In conclusion, we wish to emphasize the following points: (1) There is no sharp distinction between daughter and aunt trajectories except at $s=0$. (2) The secondary trajectories depend much more sensitively on the interaction than does the leading trajectory. As a result, the use of Lorentz symmetry to classify trajectories when $s \neq 0$ has at present uncertain practical significance. In particular, it may be dangerous to predict particles by extrapolating daughter trajectories away from $s=0$ as nearly straight lines. Also, it may be difficult to use the kinematical restrictions at $s=0$ as a guide to corrections to the leading term in particle-exchange forces. We suggest that detailed calculations will be required, in which the Lorentz symmetry is embodied in a realistic dynamical model, if we are to understand the interrelations among Regge poles.

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