POSSIBILITY OF DETECTING A SMALL TIME-REVERSAL–NONINVARIANT TERM IN THE HAMILTONIAN OF A COMPLEX SYSTEM BY MEASUREMENTS OF ENERGY-LEVEL SPACINGS

L. D. Favro and J. F. McDonald*
Department of Physics, Wayne State University, Detroit, Michigan
(Received 21 September 1967)

We have studied how a small time-reversal–noninvariant term in the Hamiltonian of a complex system affects the theoretical statistical distributions of its energy levels. We find that the major effect on the nearest-neighbor spacing distribution is in its shape near the origin. Thus, it is concluded that experimental observation of such a term by measurement of statistical distributions of energy levels is very difficult. However, one can find an upper bound to the strength of such a perturbation by such measurements.

There has been much success in describing the statistical properties of energy spectra of complex systems using ensembles of random matrices.[1] Orthogonal ensembles (which are applicable to time-reversal–invariant systems) predict spacing distributions which agree quite well with experiment.[2] In particular, such ensembles predict that the nearest-neighbor spacing distribution is linear in the spacing near the origin. Unitary ensembles (which are applicable to time-reversal–noninvariant systems) predict a quadratic dependence near the origin. A question of current interest is whether or not a small time-reversal–noninvariant part of the system Hamiltonian will manifest itself in the spacing distributions in an observable way.[3]

To investigate this we shall examine an orthogonal Gaussian ensemble of half-width $\alpha^{-1/2}$ with a random, unitary, Gaussian perturbation of half-width $\gamma^{-1/2}$. The joint distribution for the matrix elements is then

$$p(H,\gamma,\alpha)=\eta'(\gamma)\eta(\alpha)\int_{-\infty}^{\infty}\exp[-\gamma\,\mathrm{Tr}(H-H_0)^2]\times\exp[-\alpha\,\mathrm{Tr}H_0^{\,2}]dH_0, \tag{1}$$

where

$$\eta(\alpha)=2^{\frac{1}{4}N(N-1)}(\alpha/\pi)^{\frac{1}{4}N(N+1)}, \tag{2}$$

$$\eta'(\gamma)=2^{\frac{1}{2}N(N-1)}(\gamma/\pi)^{\frac{1}{2}N^2}, \tag{3}$$

and

$$dH_0=\prod_{i\geq j}(dH_0)_{ij}. \tag{4}$$

As is customary, we have chosen a representation in which the H_0 are real, and have assumed H and H_0 to be $N\times N$, with the limit of N approaching infinity to be taken at some later point.[4] The integrals over H_0 are easily performed and yield

$$p(H,\gamma,\alpha)=\frac{\eta'(\gamma)\eta(\alpha)}{\eta(\gamma+\alpha)}\exp[-\alpha'\,\mathrm{Tr}H^2]\times\exp[-2\gamma'\sum_{i>j}H_{ij}'^{2}], \tag{5}$$

where

$$\alpha'=\alpha\gamma/(\gamma+\alpha), \tag{6}$$

$$\gamma'=\gamma^2/(\gamma+\alpha), \tag{7}$$

and

$$H_{ij}'=\mathrm{Im}(H_{ij}). \tag{8}$$

The factor $\exp(-\alpha'\,\mathrm{Tr}H^2)$ describes the ordinary unitary Gaussian ensemble with a trivial change in scale. The other exponential factor yields new effects.

The joint eigenvalue distribution can be calculated exactly for $N=2$ by parametrizing the rotation matrix with the Cayley-Klein parameters. The result of this calculation is

$$p(E,\gamma,\alpha)=\alpha'(\alpha/2\pi)^{1/2}|E_1-E_2|\Phi([\tfrac{1}{2}\gamma'(E_1-E_2)^2]^{1/2})\times\exp[-\alpha'(E_1^{\,2}+E_2^{\,2})], \tag{9}$$

where Φ is the error function. The nearest-neighbor spacing distribution is easily calculated from (9) and is

$$P(S,\gamma,\alpha)=(\alpha\alpha')^{\frac{1}{2}}Se^{-\frac{1}{2}\alpha'S^2}\Phi((\tfrac{1}{2}\gamma'S^2)^{\frac{1}{2}}). \tag{10}$$

It is easily seen that this expression is approximately the orthogonal result in the region $\gamma'S^2\gg 2$, and approximately the unitary result in the region $\gamma'S^2\ll 2$. If $\alpha/\gamma\ll 1$ (i.e., the perturbation is small), the spacing distribution is quadratic in S for $S^2\ll(8\alpha/\pi\gamma)\bar{S}^2$ and linear if $8\bar{S}^2/\pi \gg S^2 \gg (8\alpha/\pi\gamma)\bar{S}^2$, where $\bar{S}$ is the average spacing for the unperturbed distribution (i.e., $\bar{S}^2 = \pi/4\alpha'$). Similar results will be shown to be valid for the N-dimensional case.

For the N-dimensional case we consider only the limit $\alpha/\gamma \ll 1$. Thus, we can use a cluster-type approximation for the last exponential factor occurring in (5).[5] To first order in small quantities,

$$\exp[-2\gamma' \sum_{i>j} H_{ij}'^2] \simeq \left(\frac{\pi}{2\gamma'}\right)^{\frac{1}{4}N(N-1)} \left[\delta(H') + \sum_{m>n} \left\{\left(\frac{2\gamma'}{\pi}\right)^{\frac{1}{2}} \exp(-2\gamma' H_{mn}'^2)\,\delta_{mn}(H') - \delta(H')\right\}\right], \tag{11}$$

where

$$\delta(H') = \prod_{i>j} \delta(H_{ij}'), \tag{12}$$

and $\delta_{mn}(H')$ has the factor for H_{mn}' missing. The terms with $\delta(H')$ obviously yield the orthogonal result [with half-width $(\alpha')^{-1/2}$]. The contribution to the joint eigenvalue distribution from terms with a delta function missing can be obtained by making an expansion in powers of $1/\gamma'$. A typical term can be written (with some multiplicative constants omitted) as

$$f_{mn} dH = (2\gamma'/\pi) \exp(-\alpha' \,\mathrm{Tr} H^2) \times \int_{-\infty}^{\infty} dx\, \delta(G') \exp(-2\gamma' x^2) dH, \tag{13}$$

where

$$G_{ij} = H_{ij} - ix(\delta_{mi}\delta_{nj} - \delta_{mj}\delta_{ni}), \tag{14}$$

and

$$dH = (\prod_{i \geq j} dH_{ij}^0)(\prod_{k>l} dH_{ij}'), \tag{15}$$

with $H_{ij}^0 = \mathrm{Re} H_{ij}$ and $G_{ij}' = \mathrm{Im} G_{ij}$. We now change variables from the H_{ij}^0 and H_i' to the eigenvalues of H and the rotation parameters φ_i which diagonalize G.[6] The result of this can be expressed in the form

$$f_{mn} dH = \frac{2\gamma'}{\pi} \exp(-\alpha' \,\mathrm{Tr} H^2) \int_{-\infty}^{\infty} dx \exp(-2\gamma' x^2)\, \delta(G') \times J(G; E', \varphi) J(E'; E)\, d\varphi\, dE, \tag{16}$$

where $J(x;y)$ is the Jacobian of the transformation from x to y and where E_i and E_i' are the eigenvalues of H and G, respectively. An expansion of $J(E';E)$ in powers of x can be obtained by treating $H-G$ as a perturbation on G and using nondegenerate perturbation theory. Such an expansion is valid if $N\gamma(E_i-E_j)^2 \gg 1$, $i>j$.

To order x^2 (i.e., to order $1/\gamma'$ in the asymptotic expansion) it can be shown that $J(E';E) \sim 1$. The form of the remaining Jacobian and resulting integration over the rotation parameters are well known since G is real, and thus corresponds to the orthogonal ensemble. Thus, to order $1/\gamma'$ the first correction term vanishes and the joint eigenvalue distribution to this order is proportional to the orthogonal Gaussian result with the scale factor α'. Hence, in the region $N\gamma' S^2 \gg 1$, the nearest-neighbor spacing distribution is, to order $1/\gamma'$, proportional to the orthogonal distribution.

The behavior of the nearest-neighbor spacing distribution near the origin (i.e., $N\gamma' S^2 \ll 1$) can be obtained by examing the formal expression for the joint-eigenvalue distribution obtained from (5). This can be written as

$$p(E,\gamma,\alpha) = \frac{\eta'(\gamma)\eta(\alpha)}{\eta(\gamma+\alpha)} \exp[-\alpha' \textstyle\sum_i E_i^2] \prod_{i>j} (E_i - E_j)^2 \times \int d\varphi\, w(\varphi) \exp[-2\gamma' \sum_{k>l} B_{kl} (E_k - E_l)^2], \tag{17}$$

where

$$B_{kl} = (\textstyle\sum a_{ik} a_{il})^2 + (\sum a_{ik} b_{il})^2. \tag{18}$$

Here a_{ij} and b_{ij} are the real and imaginary parts, respectively, of the unitary matrix which diagonalizes H. They have been parametrized in terms of some variables φ_i with weight function $w(\varphi)$. It is easily seen that the factor involving the integration over the φ is not zero and is finite at the point $S=0$, where S is some typical eigenvalue difference. Thus, the nearest-neighbor spacing distribution is to a first approximation quadratic near the origin. In conclusion, the nearest-neighbor spacing distribution is approximately the unitary result (i.e., quadratic) if $S^2 < (\alpha/\gamma)\bar{S}^2$ and approximately the orthogonal result if $S^2 > (\alpha/\gamma)\bar{S}^2$, where

$\bar{S}^2 \sim 1/N\alpha'$ is the squared average spacing for the unperturbed ensemble.

Thus, the major effect of the presence of a small time-reversal–noninvariant term on the spacing distribution is a change in shape near the origin. Since this is a region of minor probability, and hence large statistical error, it would appear that observation of such a term using statistical methods is very difficult. However, if the experimental measurements reveal that the distribution near the origin is linear for $S > S_0$, where S_0 is the smallest spacing with good statistics, one can conclude that the noninvariant term, if it exists, is at least $S_0/\bar{S}$ times smaller than the invariant term.

*Present address: Department of Mathematics, University of Windsor, Windsor, Ontario.

[1]See, for example, C. E. Porter, Statistical Theories of Spectra: Fluctuations (Academic Press, Inc., New York, 1965).

[2]See, for example, C. E. Porter and N. Rosenzweig, Ann. Acad. Sci. Fennicae: Ser. A VI No. 44 (1960).

[3]Current interest in this question was communicated to the authors by M. L. Mehta.

[4]Porter and Rosenzweig, Ref. 2.

[5]J. F. McDonald, thesis, Wayne State University, 1967 (unpublished). This method is also contained in a paper by J. F. McDonald and L. D. Favro, to be published.

[6]Here we treat G as though it were a member of a unitary ensemble. Thus, there are $N(N-1)$ parameters φ_i. The delta functions $\delta(G')$ insure that G is real (orthogonal) and $\frac{1}{2}N(N-1)$ of the φ_i integrations are trivial.

CALCULATION OF DAUGHTER TRAJECTORIES*

R. E. Cutkosky and B. B. Deo†
Department of Physics, Carnegie-Mellon University, Pittsburgh, Pennsylvania
(Received 6 October 1967)

Explicit calculations show that daughter trajectories are very model dependent. It is therefore necessary to be cautious in applying Lorentz symmetry when the energy is not zero.

At zero energy, the Regge poles obtained from the manifestly covariant Bethe-Salpeter equation occur in families; for each "mother" trajectory with a given $l(0)$, there is a sequence of "daughters" with $l_n(0) = l(0) - n$.[1-4] At $s = 0$ (s is the squared energy) the odd daughters have residues whose sign is "wrong," meaning that if the trajectories rise through physical values without this sign changing, the associated particle would be a "ghost." This sign problem is implicit in the work of Freedman and Wang[1] and had been emphasized even earlier by Nakanishi[5] (see also Ciaffaloni and Menotti[6]). An additional symmetry occurring for scalar particles of equal mass leads to several special features; in particular, the odd daughters are actually uncoupled from the scattering amplitude. It is to be expected, for reasons discussed below, that the Regge trajectories may, in the general case, behave quite differently from those previously reported for scalar particles[7] with $M_1 = M_2$ or in the nonrelativistic theory,[8] and we undertook some explicit calculations with two unequal scalar particles to demonstrate this.

We write the Bethe-Salpeter equation for an amplitude of angular mementum l in the symbolic form

$$B(l, s)\psi^l(R, \alpha) = 0, \tag{1}$$

where $B(l, s)$ is a fourth-order partial differential operator in R and α depending parametrically on l and s, R is the four-dimensional relative distance, and $t = R\cos\alpha$ is the continued[9] relative time. We assume that the interaction is a superposition of Yukawa potentials. Equation (1) is self-adjoint with the prescription

$$\psi^\dagger(R, \alpha) = \psi(R, \pi - \alpha)^*, \tag{2}$$

where the reflection $t \to -t$ makes the norm indefinite in sign and thereby leads to the possibility of ghost solutions[5] as well as to unfamiliar level-crossing behavior. We represent ψ^l as a superposition of terms of the form

$$\psi_{nk}{}^l(R, \alpha) = C_k{}^{l+1}(\cos\alpha) R^{n+l+k} e^{-\beta R} e^{\gamma R \cos\alpha} \sin^l\alpha, \tag{3}$$