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CURRENT ALGEBRA AND VERTEX FUNCTIONS*

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A method is given for calculating vertex functions involving three currents obeying chiral SU(2) ⊗ SU(2) commutation relations, without using soft-pion approximations. The procedure assumes one-particle dominance of intermediate states. The Weinberg sum rule emerges as a consequence of the vanishing of q-number Schwinger terms. The charge radius of the pion is obtained and is in agreement with existing data.

Recently Schnitzer and Weinberg[1] have developed techniques for calculating T-products of three currents obeying chiral SU(2) ⊗ SU(2) commutation relations. These results are remarkable as they allow the calculation of such processes as $\rho \to \pi + \pi$ and $A_1 \to \rho + \pi$ without resort to soft- or massless-pion approximations. Their method involves the use of the Ward identities for the vertex functions, combined with the assumption of single-particle dominance by π, A_1, and ρ mesons. We here present an alternative analysis also based on single-particle dominance, but more directly connected to the current algebras.[2] The two procedures give identical answers for the T-product of three currents but represent different techniques for extending these results to more complicated problems, such as four currents and higher groups.

We start by considering the T-product $\langle T[A_a^{\alpha}(x)A_b^{\beta}(y)V_c^{\gamma}(0)]\rangle$, where a, b, and c represent the SU(2) indices. This function may be expanded into its six time orderings of which a characteristic one is

$$\langle A_a^{\alpha}(x)V_c^{\gamma}(0)A_b^{\beta}(y)\rangle = \sum_{n,m} \langle 0|A_a^{\alpha}|n\rangle\langle n|V_c^{\gamma}|m\rangle\langle m|A_b^{\beta}|0\rangle. \tag{1}$$

Imposing single-particle dominance (i.e., saturating the right-hand side with π and A_1 states for this case) one encounters the matrix element $\langle 0|A_a^{\alpha}|\pi, qb\rangle$ which is proportional to $F_\pi q^{\alpha}$ [by partial conservation of axial-vector current (PCAC)[3]], and $\langle 0|A_a^{\alpha}|A_1, qb\rangle$ which is proportional to $g_A\epsilon^{\alpha}(q)$. Here ϵ^{α} is the polarization vector of the A_1, and g_A is the coupling strength of the axial current to the A_1 meson, defined by Weinberg.[4,3] (For other time orderings, there appears the factor $\langle 0|V_c^{\gamma} \times |\rho, qb\rangle$ which involves the coupling strength g_ρ of the vector current to the ρ meson.) For the one-meson–one-meson matrix elements, we use the single-particle–dominance hypothesis in the stronger form that the vector current couples to the mesons only through the ρ, and the axial current only through the π and A_1 particles. Thus, for example, the vertex $\langle \pi|V_c^{\gamma}|\pi\rangle$ has a ρ pole in the momentum transfer, and so one writes phenomenologically

$$(2\pi)^3(2\omega_q 2\omega_p)^{\frac{1}{2}}\langle \pi qb|V_c^{\gamma}|\pi pa\rangle = i\epsilon_{abc}\Delta_\rho^{\gamma\lambda}(k)\Gamma_\lambda(q,p) = i\epsilon_{abc}\Delta_\rho^{\gamma\lambda}(k)[a_1 + a_2k^2 + \cdots](q+p)_\lambda. \tag{2}$$

Here $\Delta_\rho^{\mu\lambda}(k)$ (where $k \equiv q-p$) is the ρ propagator and Γ_λ is the π-π-ρ vertex function. If one is not considering processes with too high a momentum transfer, presumably only a few terms in the series expansion of Γ_λ need be kept. Similarly, expressions for other one-meson matrix elements can be written down, the axial currents possessing corresponding A_1 and π poles.

In order to facilitate calculation of the T-product, it is convenient to replace the currents by field operators which are arranged to give rise to precisely the above form for the matrix elements. Since only one-particle matrix elements occur (or at most those related to them by crossing) one can replace the currents by a set of phenomenological in fields that annihilate and create the π, A_1, and ρ particles. Thus for the vacuum-one-meson matrix elements, one can replace $A_a{}^\alpha(x)$ by $g_A\tilde{a}_a{}^\alpha$ and $F_\pi \partial^\alpha\tilde{\varphi}_a$ and $V_c{}^\alpha$ by $g_\rho\tilde{v}_c{}^\alpha$ where $\tilde{a}^\alpha$, $\tilde{\varphi}_a$, and $\tilde{v}^\alpha$ are the A_1, π, and ρ in fields. For the one-meson-one-meson matrix elements, one needs a structure quadratic in the in fields. For example, for the case of Eq. (2) one can replace $V_a{}^\gamma(x)$ by

$$V_a{}^\gamma - i\epsilon_{abc}\int \Delta_\rho{}^{\gamma\lambda}(x-z) \times [a_1 - a_2\Box^2 + \cdots]\tilde{\varphi}_b(z)\partial_\lambda\tilde{\varphi}_c(z), \tag{3}$$

and similar expressions for other matrix elements. The total currents can then be replaced by the sum of all the linear plus quadratic structures needed to simulate all matrix elements that appear in the various three-point functions. One can easily verify that the assumption of single-particle dominance is satisfied then by evaluating the T-product using the equivalent total currents, but keeping only terms quartic in the in fields (in the T-product).

We have used up to now only single-particle dominance and the phenomenological form of the vertex functions. We next impose the requirement that the currents obey the SU(2)⊗SU(2) algebra. First, we note that the quadratic parts of the equivalent currents, e.g., Eq. (3), are not in general local operators. The simplest (and probably only) way of guaranteeing locality is to assume that the in-field expansions of the currents are actually solutions of field equations arising from a local Lagrangian. Thus the ρ propagator in Eq. (3) would arise naturally if $V_c{}^\gamma(x)$ were proportional to a Heisenberg ρ field $v_c{}^\gamma(x)$ (as a consequence of the Proca operator in the ρ field equations). Furthermore, the quadratic structure in Eq. (3) would occur if cubic interactions between $v_c{}^\gamma$ and a Heisenberg π field $\varphi_a(x)$ appeared in the Lagrangian. Similarly, one introduces a Heisenberg A_1 field $a_a{}^\alpha(x)$ to produce corresponding A_1 propagators in other matrix elements. The phenomenological in-field expansions such as Eq. (3) can then be simulated by writing down all possible cubic interactions between these fields, with now the guarantee that the corresponding equivalent currents are local field operators. We now choose for this equivalent Lagrangian the structure $\mathcal{L} = \mathcal{L}_{0\pi} + \mathcal{L}_{0\rho} + \mathcal{L}_{0A} + \mathcal{L}_I$, where the interaction Lagrangian is[5]

$$\mathcal{L}_I = \tfrac{1}{2}\epsilon_{abc}[h_1 v_{\mu a} v_{\nu b} G_c{}^{\nu\mu} + h_2(v_{\mu a}a_{\nu b} - v_{\nu a}a_{\mu b})H^{\nu\mu} + h_3 a_{\mu a}a_{\nu b}G_c{}^{\nu\mu} + 2f_1 v_{\mu a}\varphi_b{}^\mu\varphi_c$$
$$+ f_2\varphi_{\mu a}\varphi_{\nu b}G_c{}^{\nu\mu} + 2g_1 v_{\mu a}\varphi_b a_c{}^\mu + 2g_2\varphi_a G_b{}^{\mu\nu}H_{\mu\nu c}$$
$$+ g_3(v_{\mu a}\varphi_{\nu b} - v_{\nu a}\varphi_{\mu b})H_c{}^{\mu\nu} + g_4(a_{\mu a}\varphi_{\nu b} - a_{\nu a}\varphi_{\mu b})G_c{}^{\mu\nu}], \tag{4}$$

where h_1, h_2, etc., are a set of undetermined coupling constants. The currents are then related to the phenomenological fields by

$$A_a{}^\mu = g_A a_a{}^\mu + F_\pi\partial^\mu\varphi_a, \quad V_a{}^\mu = g_\rho v_a{}^\mu. \tag{5}$$

To calculate the T-product, one is then to solve the field equations arising from Eq. (4) and use Eq. (5) to calculate the currents. The assumption of single-particle dominance is easily seen to be equivalent to calculating the T-product to only first order in the coupling constants.

The Lagrangian of Eq. (4) is the most general cubic interaction that can be written without derivatives in a "first-order" formalism.[6] This automatically gives rise to a fixed number of terms in the momentum-transfer expansion in Eq. (2). [If additional terms in the expansion of Eq. (2) were desired, one would have to add to $\mathcal{L}_1$ additional cubic terms containing correspondingly higher derivatives.] The adoption of Eq. (4) is hence a physical assumption which is to be subjected to experimental verification.[7] We remark in passing that, from the standpoint of Lagrangian physics, Eq. (4) represents the most natural structure.

We now impose the requirements that the currents of Eq. (5) obey the chiral algebra and also that $\partial_\mu V_a{}^\mu = 0$ and $\partial_\mu A_a{}^\mu = F_\pi m_\pi{}^2 \varphi_a$. Single-particle dominance implies that these conditions be satisfied only to first order in the coupling constants h_1, h_2, etc. First, we consider the commutation relations between $\int d^3x\, A_a{}^0$, $\int d^3x\, V_a{}^0$ and $A_a{}^\mu(x)$, $V_a{}^\mu(x)$. These along with conservation of vector currents and PCAC determine (by straightforward calculation) all the coupling constants in terms of h_3 and the three parameters $x \equiv \sqrt{2}\, m_\rho/m_A$, $y \equiv g_A/g_\rho$, $z \equiv g_\rho/F_\pi\sqrt{2}\, m_\rho$. Next, we impose the more restrictive condition that the algebra obeyed by the current densities be free of q-number Schwinger terms. This produces only one further relation, the first Weinberg sum rule[4]: $x^2y^2z^2 - 2z^2 + 1 = 0$. Thus all the coupling constants are determined in terms of two of the parameters and $h_3 \equiv (m_\rho{}^2/g_\rho)\lambda_A$, where λ_A is the anomalous moment of the A_1 meson.[8] It is now straightforward to calculate the T-product of the current operators, since one need use the dynamics of Eq. (4) only to first-order perturbation theory. One finds, for example,

$$\int e^{-iqy} e^{ipx} \langle T[\partial_\alpha A_a{}^\alpha(x) \partial_\beta A_b{}^\beta(y) V_c{}^\gamma(0)]\rangle$$
$$= -F_\pi{}^2 m_\pi{}^4 g_\rho i\epsilon_{abc}(q^2+m_\pi{}^2)^{-1}(p^2+m_\pi{}^2)^{-1}\Delta_\rho{}^{\gamma\lambda}(k)[f_1(q+p)_\lambda + f_2(qkp_\lambda - kpq_\lambda)] \qquad (6)$$

in agreement with the Ward-identity analysis of Ref. 1.[9]

The current-commutator relations determine only one relation between x, y, and z. In addition to this, there is the second Weinberg sum rule,[4] based on a high-energy postulate yielding $x = 1$, and the soft-pion derivation of the result[10] $z = 1$. Since both of these derivations are somewhat less reliable than current-commutator results, it is perhaps more conservative to assume the experimentally known result $x = 1$, and search for other data to determine z or y. As pointed out by Sakurai,[11] the decay $\rho^0 \to \mu^+ + \mu^-$ gives a direct measurement of g_ρ, since the decay amplitude is proportional to $\langle 0 | V_b{}^\mu | \rho\rangle$. Assuming $z = 1$, one finds 0.37 $\times 10^{-4}$ for the muon-pair branching ratio, in satisfactory agreement with the experimental result[12] of $(0.44_{-0.09}^{+0.21}) \times 10^{-4}$. An improvement in the accuracy of this experiment could lead to a precision determination of g_ρ. Since $\rho \to \pi + \pi$ and $A_1 \to \rho + \pi$ can be calculated directly from the vertex function,[1] they yield further information on both z and λ_A. One may proceed by using the phenomenological φ_a, $a_a{}^\mu$, and $v_a{}^\mu$ as interpolating fields for the respective particles on their mass shells. If one assumes, for simplicity, $x = y = z = 1$, then both decays are reasonably consistent with a value of $\lambda_A = +0.4 \pm 0.1$. This yields ρ and A_1 widths of 114 and 93 MeV, respectively, to be compared with the experimental values[13] of 140 ± 20 and 130 ± 40 MeV. (We have used the experimental value of $F_\pi = 94$ MeV.) We note, however, that the A_1 width is sensitive to small changes in the value of $z^2 = g_\rho{}^2/(2F_\pi{}^2 m_\rho{}^2)$.

We can also apply our results to calculate the charge radius of the pion, r_e. This can be determined from the vertex $\langle \pi q | V_a{}^\mu | \pi p\rangle$. Inserting the value of $g_\rho v_a{}^\mu$ determined by Eq. (4), one obtains for the form factor (when $x = y = z = 1$)

$$f(k^2) = m_\rho{}^2(k^2 + m_\rho{}^2)^{-1}[1 + \tfrac{1}{4}\lambda_A k^2 m_\rho{}^{-2}] \qquad (7)$$

where $f(k^2) = 1 - r_e{}^2k^2/6 + \cdots$. For $\lambda_A = 0.4$ one finds $r_e = 0.6$ F, in agreement with the experimental value[14] of 0.7 ± 0.2 F. We next compare this calculation of r_e to the massless-pion value. Contracting down both pions and using PCAC in the usual fashion yields three-point and two-point parts. One obtains the identical formula for r_e if one makes the massless pion continuation $q^2 = 0 = p^2$ on the total function. The two-point part alone yields $r_e = 6^{1/2}/m_\rho$ which is the result one obtains in a ρ-dominance model. As can be seen from Eq. (7), the full result has the additional A_1 magnetic-moment term. In this case, however, ρ dominance appears to be a good approximation due to the "accidental" smallness of $\frac{1}{4}\lambda_A$.

There appear to be no essential difficulties in extending the above results to SU(3) octets of currents. The method also provides a promising approach to the calculation of T-products of four currents which would then allow the calculation of scattering amplitudes and threebody decays.

A more detailed description of the above techniques along with applications to peripheral processes will be published elsewhere.

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[2]The effective Lagrangian derived below bears a resemblance to the classical one considered by J. Schwinger, Phys. Letters 24B, 473 (1967). However, our procedure depends sensitively on the quantum nature of the operators (e.g., the presence or absence of Schwinger terms) rather than on classical considerations of gauge invariance. In spirit, our method is more closely related to the pion-nucleon effective Lagrangian of S. Weinberg, Phys. Rev. Letters 18, 188 (1967), though no assumption of soft or massless pions is made. This work was done prior to the appearance of the preprint of Ref. 1, and was stimulated by colloquia given by Weinberg and Schnitzer describing their results.

[3]We use PCAC in the form $\partial_\mu A_a{}^\mu = F_\pi m_\pi{}^2 \varphi_a$ where φ_a is the pion field. Note that our currents are normalized so as to obey equal-time commutation relations $[A_a{}^0(x),\ A_b{}^\mu(y)] = i\epsilon_{abc}V_c{}^\mu(x)\delta^3(x-y)$, etc.

[4]S. Weinberg, Phys. Rev. Letters 18, 507 (1967).

[5]In writing Eq. (4) we have used the "first-order" formalism. Thus $\varphi_a{}^\mu$ is a pion field to be varied independently of φ_a, the field equations then giving the relation between $\varphi_a{}^\mu$ and $\partial^\mu\varphi_a$. Similarly, $G_a{}^{\mu\nu} = -G_a{}^{\nu\mu}$ are the "field strengths" for the ρ field $v_a{}^\mu$, and $H_a{}^{\mu\nu}$ for the A_1 field $a_a{}^\mu$. Note that because of the presence of derivative coupling, the canonical momenta φ_{0a} differ from $\partial_0\varphi_a$ by terms linear in the coupling constants and quadratic in the fields. Similarly for $G_a{}^{\mu\nu}$ and $H_a{}^{\mu\nu}$.

[6]Alternatively, if one employed the "second-order" formalism obtained by eliminating $\varphi_a{}^\mu$ in terms of $\partial^\mu\varphi_a$, etc., Eq. (4) would then represent the most general cubic interaction containing at most first derivatives in the fields φ_a, $v_a{}^\mu$, and $a_a{}^\mu$.

[7]It is interesting to note that this condition is equivalent to the assumption of Ref. 1 that one employ in the momentum-transfer expansion the minimum number of terms consistent with the Ward identities.

[8]We find that

$$h_1 = h_2 = f_1 = m_\rho{}^2 g_\rho{}^{-1};\ g_1 = -m_A{}^2 g_3 = F_\pi m_A{}^2 m_\rho{}^2 (g_\rho g_A)^{-1};$$

$$g_\rho f_2 = [1 - g_A{}^2(F_\pi{}^2 m_A{}^2)^{-1}] + g_A{}^2 m_\rho{}^2 (F_\pi{}^2 m_A{}^4)^{-1}\lambda_A;$$

$$g_4 = (g_\rho F_\pi)^{-1} g_A(\lambda_A m_\rho{}^2 m_A{}^{-2} - 1);$$

and

$$2F_\pi g_2 = g_A g_\rho{}^{-1} - g_\rho g_A{}^{-1}.$$

[9]While q-number Schwinger terms have been eliminated from the current commutators, it is still possible for noncovariant Schwinger terms to arise in the T-products (as is the case for two current operators). Such structures would posses one less pole than the normal terms, and hence not contribute to mass-shell calculations but conceivably could effect off-shell results as in vertex form factors. Actually, all such Schwinger terms cancel in the T-products of three current operators, which tends to justify the use of these results off the mass shell. However, Schwinger terms are present in some of the field T-products, e.g., $\langle T(\partial_\alpha A_a{}^\alpha a_b{}^\beta V_c{}^\mu)\rangle$.

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