SUM RULE FOR RESONANCE REACTIONS*

P. A. Moldauer
Argonne National Laboratory, Argonne, Illinois
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It is shown that in the absence of direct reactions, the averages of the diagonal elements of the pole-residue matrices $-i\mathsf{G}_\mu$ of the S matrix (sometimes called partial widths) satisfy

$$2\pi\langle G_{\mu cc}\rangle_\mu/D=\exp(2i\xi_c)T_c/(1-T_c)^{1/2},$$

where D is the average pole spacing, T_c is the optical-model transmission coefficient, and ξ_c is the real part of the optical-model phase shift.

Result.—The S matrix for resonance reactions is most conveniently represented by the resonance-pole expansion

$$\mathsf{S}=\mathsf{S}^0-i\textstyle\sum_\mu \mathsf{G}_\mu/(E-\mathcal{E}_\mu), \tag{1}$$

whose energy dependence is specified by the constant pole coordinates $\mathcal{E}_\mu$, and by the background matrix S^0 and the residue matrices G_μ, each of which has branch points associated with thresholds. The representation (1) permits the simple calculation of cross sections which are proportional to the absolute squares of the matrix elements of S. Unfortunately, however, the parameters needed to specify S in the form of Eq. (1) are not independent dynamical variables but rather satisfy complicated relationships which arise chiefly from the requirement that S be unitary. In previous work the nature of those relationships was explored by means of models of Eq. (1) having relatively simple analytic forms[1] and by means of numerical calculations.[2,3] A useful way of stating such results is to relate sums or averages of the parameters in Eq. (1) to the elements of the average S matrix $\langle\mathsf{S}\rangle$. In the absence of direct reactions, $\langle\mathsf{S}\rangle$ is diagonal, and its elements are by definition the optical-model scattering functions for the various open channels.

In the present work we derive a sum-rule relation between the diagonal elements $G_{\mu cc}$ of the residue matrix, summed over the resonance index μ, and the corresponding element $\langle S_{cc}\rangle$ of the average S matrix, when the latter is diagonal. The result can be stated in the form

$$2\pi\langle G_{\mu cc}\rangle_\mu/D=2\sinh(2\eta_c)\exp(2i\xi_c), \tag{2}$$

where

$$\langle S_{cc}\rangle=\exp[2i(\xi_c+i\eta_c)]. \tag{3}$$

The bracket $\langle\ \rangle_\mu$ indicates an average over μ and D is the average spacing of the real parts of the pole positions $\mathcal{E}_\mu$. The complex number $\xi_c+i\eta_c$ is the optical-model phase shift for the alternative c in the energy region of the resonances μ that are included in the average of Eq. (2).

The result of Eq. (2) has important applications in the calculation of average cross sections.[1] It also provides a lower bound for an even more important parameter:

$$2\pi\langle|G_{\mu cc}|\rangle_\mu/D\geq T_c/(1-T_c)^{1/2}. \tag{4}$$

Here we have used the definition of the optical-model transmission coefficient

$$T_c=1-\exp(-4\eta_c). \tag{5}$$

But perhaps the greatest significance of Eq. (2) lies in the fact that it provides a generally valid confirmation for the previously discovered behavior of resonance parameters in the strong-absorption limit $\eta_c\to\infty$.[1-3] In that limit $\bar{S}_{cc}\to 0$ and T_c goes to unity. The relationship

$$2\sinh 2\eta_c=T_c/(1-T_c)^{1/2} \tag{6}$$

shows clearly that the left-hand sides of Eqs. (2) and (4) diverge as $T_c\to 1$ and that therefore no generally applicable upper bound limits the possible values of $2\pi\langle G_{\mu cc}\rangle_\mu/D$ or $2\pi\langle|G_{\mu cc}|\rangle_\mu/D$. Heretofore it has been widely assumed in the literature on resonance reactions that the above parameters cannot exceed the value of unity (or, in a few instances, the value of 2.0). The origins of these misconceptions have been discussed in Ref. 1. By substituting Eq. (6) into Eq. (2) it is easily verified that all the special models of Ref. 1 satisify the sum rule (2).

Proof.—To prove Eq. (2) we consider all the

poles μ of S whose $\mathrm{Re}\mathcal{E}_\mu$ lie in an energy interval ΔE centered on E_0 and containing no thresholds and (by choice) no branch cuts. We calculate the sum of the residues G_μ of these poles by integrating S along a closed contour surrounding the $\mathcal{E}_\mu$. We choose for this contour a rectangle of width ΔE in the direction of the real axis and of height $2W$ in the direction of the imaginary axis, with its center at E_0 on the real axis. We assume that the values of the widths $\Gamma_\mu = -\mathrm{Im}\mathcal{E}_\mu$ are bounded so that we can choose W to be large compared with this bound. Then S will be essentially constant along the top and bottom horizontal legs of the contour and

$$2\pi\sum_\mu G_\mu = \Delta E[S(E_0 - iW) - S(E_0 + iW)] + \Delta, \quad (7)$$

where Δ is the contribution of the two vertical legs of the contour.

Identifying $S(E_0 + iW)$ with $\langle S\rangle$,[4] and using the fact that because of analyticity and unitarity[5]

$$S(E^*) = S^{*-1}(E), \quad (8)$$

we find that

$$2\pi\sum_\mu G_\mu = \Delta E(\langle S^*\rangle^{-1} - \langle S\rangle) + \Delta. \quad (9)$$

On the average (over many different sets of poles $\mathcal{E}_\mu$) the contribution Δ is expected to vanish. This, together with the requirement that $\langle S\rangle$ be diagonal (no direct reactions), and the definition (3), immediately leads to the result of Eq. (2). Moreover, the application of perturbation theory to Eq. (9) shows that small direct-reaction amplitudes affect the result (2) only by terms which are of the second order in the ratios of the average off diagonal to the average diagonal S-matrix elements.

Discussion.—We note that our sum rule (2) is a sum rule "on the average" only. This is as it must be. If there existed any such sum rule that held exactly for N resonance poles, it would also have to hold exactly for $N+1$ poles, in order to be of any practical use. But a relationship that holds exactly for both a sum over parameters of any N and $N+1$ poles must hold exactly for the parameters of each pole separately. That no fixed relationship exists between pole parameters, say between the residue and the width, of a single pole is easily verified by considering the example of two overlapping resonances in any one of the available unitary reaction formalisms.

The statement that Δ vanishes on the average depends on the assumption that $\langle S\rangle$ does not vary with energy. The definition of $\langle S\rangle$, and therefore its energy variation, requires the specification of an averaging interval and a resolution function. Our choice $S(E_0 + iW) = \langle S(E_0)\rangle$ is the average obtained with a Lorentzian resolution function of width W. With a rectangular resolution function one obtains $S(E_0 + iW) = (1 + iWd/dE_0)\langle S(E_0)\rangle$. On the other hand, with an energy-dependent $\langle S\rangle$ we have on the average $\Delta(E_0)/\Delta E \cong 2Wd\langle S(E_0)\rangle/dE_0$. We see therefore that the error in Eq. (2) due to the neglect of Δ is of the same order as the common uncertainty in the average because of the ambiguity in its definition.

As seen from the results of Refs. 1-3, the values of the widths Γ_μ also tend to infinity as $\eta \to \infty$. This means that in some situations our requirement that $W \gg \Gamma_\mu$ cannot be satisfied in practice because some or all widths are as large as or larger than any useful averaging interval. At what point this occurs will depend on the particular situation; that is, on the transmission coefficients of all competing channels, on the proximity of thresholds, etc. In such cases one can adopt one of two attitudes. Either one can say that the process of averaging has lost its usefulness, or one can compare averages with the expectation values predicted by the method of the statistical S matrix.[1]

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[1]P. A. Moldauer, Phys. Rev. 157, 907 (1967).

[2]P. A. Moldauer, Bull. Am. Phys. Soc. 12, 27 (1967), and to be published.

[3]P. A. Moldauer, Phys. Rev. Letters, 18, 249 (1967).

[4]P. A. Moldauer, Phys. Rev. 129, 754 (1963).

[5]A. M. Lane and R. G. Thomas, Rev. Mod. Phys. 30, 257 (1958). The property (8) is also easily verified by inspection of the R-matrix formulation of the S matrix.