problem has been discussed by Yang and Yang.[8] In their notation the Hamiltonian is

$$H(\Delta) = -\tfrac{1}{2}\sum\{\sigma_x\sigma_x' + \sigma_y\sigma_y' + \Delta\sigma_z\sigma_z'\}. \tag{2}$$

Δ measures the anisotropy, and we find the correspondence

$$2\Delta = e^{-\delta/T} + e^{+\delta/T} - e^{2\epsilon/T}. \tag{3}$$

The transformation

$$\delta \to \delta' = -\delta \tag{4}$$

leaves Δ invariant, as it should. From now on we may restrict ourselves to $\delta \geq 0$, or if $\eta = e^{\delta/T}$, then $\eta \geq 1$.

The eigenvalue $\Lambda(y)$ is the sum of two terms,

$$\Lambda(y) = \Lambda_R(y) + \Lambda_L(y). \tag{5}$$

For $\Delta \leq 1$, the two terms may be written

$$\Lambda_R(y) = e^{N\delta/2T} \prod_{j=1}^{n} \left\{\frac{2\Delta - e^{\delta/T} - \exp(ik_j)}{e^{\delta/T} - \exp(ik_j)}\right\}, \tag{6}$$

$$\Lambda_L(y) = \Lambda_R(y) \text{ with } \delta \to -\delta. \tag{7}$$

The k_j's are as given in Yang and Yang; they become distributed with a density $\rho(k)$ in the limit $N \to \infty$. The free energy per site F, with applied field, is

$$F = \min_y\{-Ey - (T/N)\ln[\max(\Lambda_R(y), \Lambda_L(y))]\}. \tag{8}$$

In the limit $N \to \infty$,

$$\frac{\ln\Lambda_R(y)}{N} = \left(\frac{1+|y|}{4}\right)\frac{\delta}{T} + \frac{1}{2}\int \rho(k)dk \ln\left[\frac{(2\Delta - e^{\delta/T})^2 - 2(2\Delta - e^{\delta/T})\cos k + 1}{2(\cosh\delta - \cos k)}\right]. \tag{9}$$

We then find that for $\Delta \leq 1$, $\Lambda_R(y) \geq \Lambda_L(y)$.

The singularities in the thermodynamic functions occur in y at $y = 0$, $\Delta < 1$; and in T at $\Delta = -1$, $y = 0$ and $\Delta = +1$, all y.

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Note added in proof.—We recently received a preprint from E. H. Lieb, treating the Rys F model in a similar way with similar results.

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UNIVERSAL INSTABILITY OF A RESISTIVE PLASMA COLUMN*

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The stability of a collision-dominated, weakly ionized plasma column confined in a magnetic field has been considered by a number of authors,[1-3] and their results have been substantiated by experimental observations of instabilities in positive column-type discharges. These instabilities are caused by differential mobility of the charged particles in the applied or self-generated equilibrium electric fields. Our concern here is with a more prevalent universal mode, related to some previously derived drift modes, that also grows via differential drifts but sets in when the column is somewhat more highly ionized. Thus the model applies to such low-temperature laboratory plasmas as those generated by a differentially pumped hollow-cathode discharge.

We have carried out an analysis for such a highly ionized two-fluid plasma and found that the growth rate of the modes driven by the equilibrium electric field indeed decreases; then the more prevalent universal mode dominates. A related prior analysis by Galeev, Oraevskii, and Sagdeev[4] has shown that such a column is unstable in the presence of longitudinal thermal conductivity. The instability is, in fact, more general and will occur for any longitudinal electron motion either from density or temperature gradients.

Our analysis is based on the usual linearized two-fluid model[2] with finite resistance due to presence of some neutral particles. The equations are written for Cartesian slab geometry. For a cylindrical plasma column (the configuration later described in Fig. 1), we associate $r\equiv x$, $r\theta\equiv y$, $z=z$. This association is also customary, and saves some algebraic complexity. By combining the continuity and momentum equations, we obtain two equations for the perturbed density and potential functions of the form

$$n_i(x,y,z,t)=n_e(x,y,z,t)$$
$$=n(x)\exp[i(\gamma y+kz-\omega t)], \qquad (1)$$
$$\varphi(x,y,z,t)=\varphi(x)\exp[i(\gamma y+kz-\omega t)]. \qquad (2)$$

After standard but tedious reductions, we obtain two simultaneous differential equations for the perturbations n_i and n_e. In these, we replace the differential gradient operator $\partial/\partial x$ $\equiv\partial/\partial r$ by $1/p$, where p is a characteristic length over which the plasma density changes considerably. Here $p\leq x_0$, where x_0 is the x dimension of the plasma, or the radius if the plasma is a cylindrical column. The determinant of the two (now) algebraic equations is set equal to 0 for nontrivial solutions, yielding a dispersion relation among ω, k, x_0, p, etc. The condition $\mathrm{Im}(\omega)=0$ determines the final stability criterion, and we write it as follows:

$$(\beta_e+\beta_i)(\delta_e\beta_i+\delta_i\beta_e)+(\eta_e-\eta_i)(\mu_i\beta_e+\mu_e\beta_i)$$
$$-(\mu_e-\mu_i)(\delta_e\mu_i-\delta_i\mu_e)>0 \text{ (stable)}. \qquad (3)$$

Here, we have defined

$$\delta_e=D_{e\perp}(\pi^2/x_0^{\,2}+\gamma^2)+k^2D_e,$$
$$\delta_i=D_{i\perp}(\pi^2/x_0^{\,2}+\gamma^2)+k^2D_i,$$
$$\beta_e=[b_{e\perp}(\pi^2/x_0^{\,2}+\gamma^2+1/2p^2)+k^2b_e]T_e,$$
$$\beta_i=[b_{i\perp}(\pi^2/x_0^{\,2}+\gamma^2+1/2p^2)+k^2b_i]T_e,$$
$$\eta_e=\gamma b_{e\mathrm{H}}E_x+kb_eE_z,$$
$$\eta_i=\gamma b_{i\mathrm{H}}E_x-kb_iE_z,$$
$$\mu_e=(\gamma/p)b_{e\mathrm{H}}T_e,$$
$$\mu_i=(\gamma/p)b_{i\mathrm{H}}T_e,$$

in which D and b are the diffusion and mobility coefficients, and T is the temperature. The subscripts e and i refer to the electrons and ions, respectively, and the symbols $\perp$ and H designate the perpendicular and Hall directions with respect to the applied magnetic field.

The stability criterion contains three terms. The first is positive definite and represents dissipation effects which are always stabilizing. The second term contains the effects arising from differential drift (the term $\eta_e-\eta_i$) of the two fluids in equilibrium electric fields. Since that term will be negative for some perturbations, those effects always tend to destabilize and lead to the modes discussed by Kadomtsev and Nedospasov[1] for longitudinal electric fields, and by Hoh[2] and Simon[3] for transverse electric fields. Making the proper simplifications, we find the same stability criteria derived by those authors from Eq. (3). Our concern here is with effects arising from the differential drifts $(\mu_e-\mu_i)$ given by the third

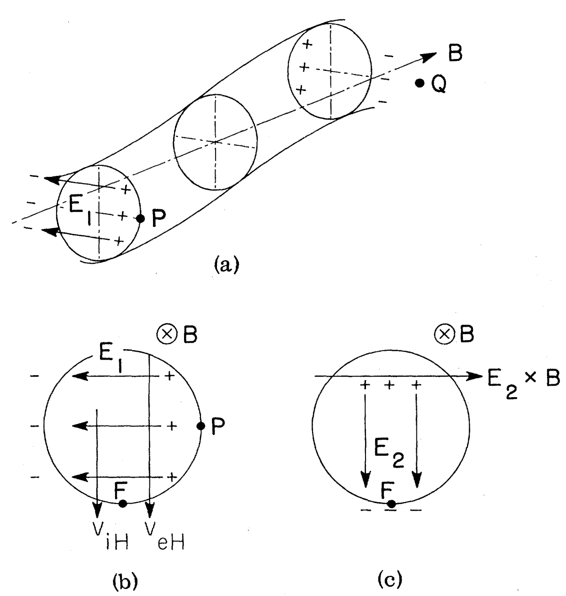

FIG. 1. Axial-diffusion-driven instability.

term in Eq. (3).

For very weak magnetic fields (i.e., the product of the ion cyclotron frequency and the ion collision time is much less than unity, corresponding to the cases considered by the authors in Refs. 1-3), it is stabilizing. However, as the effect of the magnetic field begins to be felt by the ions, we achieve the state where $\delta_e\mu_i > \delta_i\mu_e$ and the differential drift is destabilizing. In such cases, the ratio of the third to the second terms in Eq. (3) is

$$\frac{(\mu_e-\mu_i)(\delta_e\mu_i-\delta_i\mu_e)}{(\eta_e-\eta_i)(\mu_i\beta_e+\mu_e\beta_i)}\approx\frac{T_e}{T_i}, \tag{4}$$

and therefore this differential-drift destabilizing effect is dominant in plasmas in which $T_e \gg T_i$, i.e., in most differentially pumped discharges.

The mechanism of the instability can be understood physically by considering the model shown in Fig. 1. A radially inhomogeneous plasma column of density n_0 develops some helical or kink perturbation as shown in Fig. 1(a). The density is then nonuniform along the magnetic field lines, being higher at point P than at point Q. The electrons diffuse easily along the field lines, tending to set up a new equilibrium with point P positive and point Q negative. Further development of the instability is shown at the cross section through P, in Figs. 1(b) and 1(c). A potential difference now exists across the plasma, producing the electric field E. For the magnetic field B as shown, Hall mobilities $\mu_{e\mathrm{H}}$ and $\mu_{i\mathrm{H}}$ drive electrons and ions downward; but $\mu_{e\mathrm{H}} > \mu_{i\mathrm{H}}$ for a finite-resistivity plasma column, and a second space-charge separation develops as in Fig. 1(c). The resulting second electric field component E_2 causes a second Hall flow in the $\vec{E}_2\times\vec{B}$ direction—in the direction of the original perturbation. We see that this instability will be easy to excite, being driven (in part) and not stabilized by axial electron diffusion. The effect disappears in the limit of no electron- or ion-neutral collisions. Then the Hall drift velocity of the electrons and ions becomes equal, and the validity of our partially ionized plasma model breaks down also.

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OPTICAL BIREFRINGENCE AND CRYSTAL GROWTH OF HEXAGONAL-CLOSE-PACKED He^4 FROM SUPERFLUID HELIUM*

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The hexagonal-close-packed (hcp) phase of solid He^4 has exhibited anisotropy in sound-velocity measurements[1,2] and thermal-conductivity measurements.[3] Interpretation of these data requires a knowledge of crystal orientation. Optical birefringence can be used to determine the symmetry axis of hexagonal crystals since the optic axis coincides with the symmetry axis (c axis). We have made measurements of the difference in the indices of refraction of extraordinary and ordinary light in solid hcp He^4 at temperatures between 1.2 and 1.4°K and pressures between 25.0 and 26.0 atm. We obtain a value $|n_e-n_o| = (2.6\pm0.1)\times10^{-6}$. A quarter-wave plate for hcp He^4 at a wavelength of 6328 Å is thus about 6 cm thick.

The sample cells are cylindrical Pyrex glass chambers with flat end windows. Two different cells are used in the experiment, one 5.8 cm in length and the second 2.5 cm long. The solid is formed by slowly increasing the pressure on the liquid helium in the cell, which is completely immersed in the outer helium bath at about 1.2°K, until the freezing pressure is reached. Pressure is applied through vacuum-jacketed capillary tubing provided with appropriate heaters to keep the capillary tubing from blocking with solid. Large crystals (on the order of 2 cm^3) can be grown in less than 5 min using this technique. Below we present evidence