DISPERSION RELATIONS AND THE PROTON-NEUTRON MASS DIFFERENCE

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This Letter is written with reference to a paper[1] written by one of us, hereafter referred to as I, and to a paper by Harari.[2]

In I a sum rule was derived for the proton-neutron mass difference as calculated in second-order perturbation theory. The derivation depended on being able to write unsubtracted dispersion relations for the forward virtual photon Compton scattering amplitude. Harari has argued that one of the dispersion relations in I does in fact require a subtraction. We argue here that if a subtraction is needed in the dispersion relations then the usual second-order perturbation theory will diverge.

If second-order perturbation theory in the electric charge e does give a finite answer for the proton-neutron mass difference then this mass difference is

$$\Delta M=\frac{i}{8\pi^2}\int\frac{T(\vec{q},q^0)d^4q}{\vec{q}^2-(q^0)^2-i\epsilon}, \tag{1}$$

where

$$T(\vec{q},q^0)=2i\pi\int e^{-i\vec{q}\cdot\vec{x}+iq^0x^0}\tilde{T}(x)\theta(x^0)d^4x,$$
$$\tilde{T}(x)=[\langle p|+\langle n|]\{[j_\mu(x),j_\nu(0)]\}[|p\rangle-|n\rangle]g^{\mu\nu};$$

$|p\rangle$ and $|n\rangle$ are the proton and the neutron state at rest and $j_\mu(x)$ is the electromagnetic current operator.

It was shown in I that, by turning the q^0 integration from the real to the imaginary axis, ΔM could be written as

$$\Delta M=-\frac{1}{4\pi}\int_0^\infty\frac{H(q^2)}{q^2}dq^2, \tag{2}$$

$$H(q^2)=\int_{-q}^{+q}T(q^2,iq^0)[q^2-(q^0)^2]^{1/2}dq^0,$$
$$q^2=\vec{q}^2+(q^0)^2, \tag{3}$$

where we have made the change of variable $q^0\to iq^0$.

Assuming the postulates of local field theory, we have the Jost-Lehmann representation for T. Because of the spherical symmetry of T this can be written, apart from substractions, as

$$T(\vec{q},q^0)=\frac{1}{|\vec{q}|}\int\rho(x,v)\ln\left[\frac{v+q^2+2x|\vec{q}|}{v+q^2-2x|\vec{q}|}\right]dxdv, \tag{4}$$

where the weight function $\rho(x,v)$ is real and is nonzero only over the region

$$0\leq x\leq M,\quad M=\text{nucleon mass};$$
$$v-2M[M-(M^2-x^2)^{1/2}]\geq 0.$$

For $q^2>0$ we have, from Eq. (4), that the imaginary part of T satisfies

$$\operatorname{Im}T=(\pi/|\vec{q}|)\int_R\rho(x,v)dxdv, \tag{5}$$

where R is the region $v+q^2-2x|\vec{q}|\leq 0$.

We also have from I

$$\operatorname{Im}T(q^2,w)=(2\pi)^{-2}\Sigma(q^2,w),$$
$$\Sigma(q^2,w)=\Sigma^p(q^2,w)-\Sigma^n(q^2,w), \tag{6}$$

where we have introduced the new variable

$$w=q^0-q^2/2M$$

which makes the nucleon pole position and the inelastic threshold independent of q^2; p and n refer to the proton and neutron state, respectively;

$$\Sigma^p(q^2,w)=w[2\sigma_T^{\ p}(q^2,w)-\sigma_L^{\ p}(q^2,w)]$$
$$+\pi e^2\delta(w)\left[\frac{q^2}{2M}[G_M^{\ p}(q^2)]^2-[G_E^{\ p}(q^2)]^2\right]. \tag{7}$$

$\delta(w)$ is the Dirac δ function, σ_T and σ_L are the transverse and longitudinal virtual photon total cross sections as defined, for example, by Hand.[3] G_M and G_E are the magnetic and electric nucleon form factors.

From Eq. (4), if $\rho(x,v)$ decreases fast enough as $v\to\infty$ we can write unsubtracted dispersion relations for T at fixed $q^2>0$:

$$T(q^2,w)=\frac{1}{4\pi^3}\int_{0-}^\infty\Sigma(q^2,w')\left[\frac{1}{w'-w}+\frac{1}{w'+w+q^2/M}\right]dw'. \tag{8}$$

Inserting this into Eq. (3) we get

$$H(q^2)=\frac{1}{2\pi^2}\int_{0-}^\infty\left[\left\{\left(w+\frac{q^2}{2M}\right)^2+q^2\right\}^{1/2}-\left(w+\frac{q^2}{2M}\right)\right]$$
$$\times\Sigma(q^2,w)dw. \tag{9}$$

This expression for $H(q^2)$ with Eq. (2) gives a similar sum rule to that obtained in I. This sum rule could also have been obtained directly from Eqs. (4)-(6) without the intermediate step of the fixed-q^2 dispersion relations.

According to Harari, $\Sigma(q^2, w)$ will not decrease fast enough as $w \to \infty$ to make the dispersion relation, Eq. (8), converge. Harai suggests, in fact, that $\Sigma(q^2, w) \sim w^{0.4}$ for large w. This would imply that one subtraction is necessary in this dispersion relation and also in the representation Eq. (4). A subtraction in the Jost-Lehmann representation can be made at one point in $(q^2, (q^0)^2)$ space[4]—for example, at the point $q^2 = -(q^0)^2 = +\epsilon^2$. We then have

$$T(\vec{q}, q^0) = C + \int \rho(x, v)\left[\frac{1}{|\vec{q}|}\ln\left[\frac{v + q^2 + 2x|\vec{q}|}{v + q^2 - 2x|\vec{q}|}\right] - \frac{4x}{v + \epsilon^2}\right] dx\, dv. \tag{10}$$

This is equivalent to adding an infinite constant

$$C - \int \rho(x, v)\left(\frac{4x}{v + \epsilon^2}\right) dx\, dv = C - \frac{1}{2\pi^3}\int \Sigma(\epsilon^2, w')\left[\frac{w' + \epsilon^2/2M}{(w' + \epsilon^2/2M)^2 + \epsilon^2}\right] dw' \tag{11}$$

into Eq. (4) to compensate for the infinite integral. We can, if we want, take the limit as $\epsilon \to 0$. The value of C is then determined from the known Thomson limit to be $C = -(3/M)(e/2\pi)^2$. If T is well defined by Eq. (10) then $H(q^2)$ is well defined by Eq. (3). Putting in the compensating infinite quantities of Eq. (10) separately gives the formula

$$H(q^2) = -\frac{3e^2q^2}{8\pi M} + \frac{1}{(2\pi)^2}\lim_{\epsilon \to 0}\int_{0^-}^{\infty}\left\{2\left[\left\{\left(w + \frac{q^2}{2M}\right)^2 + q^2\right\}^{1/2} - \left(w + \frac{q^2}{2M}\right)\right]\Sigma(q^2, w) - q^2\left[\frac{w + \epsilon^2/2M}{w^2 + \epsilon^2}\right]\Sigma(0, w)\right\} dw. \tag{12}$$

We have taken the limit as $\epsilon \to 0$. The justification of this manipulation with infinite quantities involves working directly with the Jost-Lehmann representation. A full discussion of this point will be given in a later paper.

In order that $H(q)^2$ be well defined, $\Sigma(q^2, w) - \Sigma(0, w)$ must tend to zero or be an oscillating function of w for large w. Suppose that, in fact, this expression does tend to zero. This implies that when a high-energy "virtual photon" strikes a nucleon the processes involved are insensitive to an exact energy-momentum balance so that there is little distinction between real and "virtual" photons. This seems to us to be a reasonable property to be expected of these cross sections, since at high energies the structure of most of the final states involved will be very complicated.

What now about the convergence of our final integral over q^2, Eq. (2)? For $q^2 \gg 4M^2$,

$$\frac{H(q^2)}{q^2} \simeq \frac{3e^2}{8\pi M} + \frac{1}{(2\pi)^2}\lim_{\epsilon \to 0}\int_{0^-}^{\infty}\left[\frac{\Sigma(q^2, w)}{w + q^2/2M} - \left(\frac{w + \epsilon^2/2M}{w^2 + \epsilon^2}\right)\Sigma(0, w)\right] dw. \tag{13}$$

Experimentally, for fixed w in the low-energy region, $\Sigma(q^2, w)$, in magnitude, is a decreasing function of q^2. If this is a general feature of these cross sections and if a subtraction is necessary in the dispersion relations then $(1/q^2)H(q^2) \to \infty$ as $q^2 \to \infty$. Unless some very delicate cancellations are taking place, for which we can see no a priori reason, the integral in Eq. (2) diverges.

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[4]R. Jost and H. Lehmann, Nuovo Cimento 5, 1598 (1957); F. J. Dyson, Phys. Rev. 110, 1460 (1958).