for $\lambda'-\lambda$ odd. There are at least $\frac{1}{2}(2S_m+1)$ such amplitudes where S_m is the minimum of (S_f, S_i). Also there are $\frac{1}{2}(2S_m+1)$ independent matrix elements,

$$\begin{pmatrix} 27 & 8 & 8 \\ \Sigma & \alpha & \beta \end{pmatrix} \langle S_f | \Gamma_\rho^{\alpha +} \Gamma_\rho^{\beta} | S_i \rangle .$$

Thus we expect

$$\begin{pmatrix} 27 & 8 & 8 \\ \Sigma & \alpha & \beta \end{pmatrix} \Gamma_\rho^{\alpha +} \Gamma_\rho^{\beta} = 0 \tag{12}$$

as an operator equation. We see that the sum rule resulting from crossing to the $\underline{27}$ amplitude in the t channel is satisfied by exhibiting a set of octet operators, for each helicity λ considered, whose product $\Gamma_\lambda^{\alpha +}\Gamma_\lambda^{\beta}$ contains no $\underline{27}$ part.

A solution to (12) is obtained if one assumes that Γ^β with the generators G^α of SU(3) form the group $\mathrm{SU}(3)_1 \otimes \mathrm{SU}(3)_2$. Let $G^\alpha = G_1^\alpha + G_2^\alpha$, where G_1^α and G_2^α are the generators of the two SU(3) groups. Let $\Gamma^\beta = G_1$. Then (12) is satisfied if one restricts the representations of $\mathrm{SU}(3)\otimes\mathrm{SU}(3)$, which are characterized by a representation of $\mathrm{SU}(3)_1$ and $\mathrm{SU}(3)_2$, so that the representation of $\mathrm{SU}(3)_1$ does not contain a $\underline{27}$ in the product with its adjoint—for instance, a $(\underline{3}, \underline{6})$.

We have not obtained a Lie-group structure as we might have hoped. However, we have reduced the coupled superconvergent relation to an operator equation for each helicity.

The details of these considerations will be published elsewhere.

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†Alfred P. Sloan Fellow.

[1] V. DeAlfaro, S. Fubini, G. Rossetti, and G. Furlan Phys. Letters 21, 576 (1966); L. D. Soloviev, Dubna Report No. E2343, 1965 (unpublished); R. Oehme, Phys. Letters 21, 567 (1966).

[2] B. Sakita and K. C. Wali, Phys. Rev. Letters 18, 31 (1967); P. Babu, F. J. Gilman, and M. Suzuki, Phys. Letters 24B, 65 (1967); G. Altarelli, F. Buccella, and R. Gatto, Phys. Letters 24B, 57 (1967); V. A. Matveev, B. N. Struminsky, and A. N. Tavkhelidze, Phys. Letters 23, 146 (1966).

[3] T. Cook, C. J. Goebel, and B. Sakita, Phys. Rev. Letters 15, 35 (1965).

[4] Note added in proof:—While this paper was being written, L. K. Pande published an argument showing that the assumption of saturation in the static model leads to the same result as the strong-coupling model; L. K. Pande, Phys. Letters 24B (1967).

[5] L. Durand, P. C. DeCelles, and R. B. Marr, Phys. Rev. 126, 1882 (1962).

[6] T. L. Trueman, Phys. Rev. Letters 17, 1198 (1966).

[7] L. C. Wang, Phys. Rev. 142, 1187 (1966).

"TRY SIMPLEST CASES" DISCOVERY OF "HIDDEN MOMENTUM" FORCES ON "MAGNETIC CURRENTS"

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Formulations of electromagnetic theory which include magnetic currents[1] carried by magnetic charges predict a force density f_m analogous to the Lorentz force. Consider a medium characterized in mks notation by ϵ and σ and a magnetic charge current density $\mu_0\dot{M}$ so that

$$\nabla \times H = \epsilon\dot{E} + \sigma E = \dot{D} + J, \tag{1}$$

$$-\nabla \times E = \mu_0\dot{H} + \mu_0\dot{M} = \dot{B}. \tag{2}$$

(Scalars and vectors are mostly indicated by context rather than symbols.) The expected force per unit volume on inertial particles (i.e., matter) is $f_e + f_m$, where

$$f_e = (\dot{D} - \epsilon_0\dot{E} + J) \times \mu_0 H, \tag{3}$$

$$f_m = \epsilon_0 E \times \mu_0\dot{M} = \epsilon_0 E \times (\dot{B} - \mu_0\dot{H}). \tag{4}$$

From our analysis we conclude that f_m will apply not only to hypothetical[1] currents of magnetic charges but also to the real case[2] in which $\mu_0\dot{M}$ is caused by Amperian current loops such as quantum mechanical spin or orbital motions;

in the latter case f_m is a "pseudoforce" in the sense that it may appear as acceleration of the center of mass of a collection of matter while the total momentum of the collection remains constant. The origin of f_m is conversion of an apparently hitherto neglected "hidden momentum" $G_l = -\epsilon_0 E \times \mu_0 m$ associated with energy flow in a current loop of magnetic dipole strength m situated in an electric field E. This energy-flow momentum is required in steady-state conditions to prevent local changes in mass density. Conversion of this "hidden momentum" to other forms may have the effect of a force accelerating the center of mass of the matter involved.

The need for f_m in addition to f_e was recognized by applying the "try simplest cases" search thinking tool[3] to a superconducting coaxial line with a thin annular space ($W \ll$ inner circumference, $2\pi R$) and annular area $A = 2\pi RW$ that carries power at rate VI and transports mass a distance L from an input to an output battery so that the total electromagnetic momentum is

$$G_p = VIL/c^2 = g_p AL = (E \times H/c^2)AL, \tag{5}$$

leading to the conclusion that the electromagnetic momentum density is

$$g_p = E \times H/c^2 \tag{6}$$

even when ϵ and μ in the annular space do not equal ϵ_0 and μ_0.[5,7]

The "conceptual experiment"[3] of introducing resistance on the center conductor causes G_P to be converted to linear momentum of the coaxial line and f_m is found to be required to conserve momentum.

The mystery of the force f_m is made vivid with the "idealized limiting case"[3] shown in Fig. 1. Two charged spheres are supported from a pill box by rods. The pill box contains a magnetic dipole $m = IA$ polarized in the z direction, the current being produced by a symmetrical pair of counter-rotating disks with charged rims. Initially, the entire assembly sits with its center of mass at rest at the origin of the coordinate system while the disks rotate without friction. Next, a vanishingly small frictional force is imagined to bring the disks so slowly to rest that radiation is negligible, thus producing field $E_\theta = -\mu_0 \dot{I}A/4\pi X^2$. The two charges are given a combined total impulse in the $-y$ direction,

$$G_Q = -\hat{j}Q\mu_0 IA/2\pi X^2 \tag{7}$$

but have negligible velocity if M/Q is arbitrarily large. Is momentum conserved? Is there a force on the disks so that they acquire a compensating impulse $G_D = -G_Q$?

The answer is understood in terms of G_l, an apparently hitherto disregarded momentum[6,7] as shown in Fig. 2. Figure 2(a) shows the g_p distribution outside the pill box. Since E is nearly parallel to the x axis, the g_p lines are approximately in y-z planes. Those that enter the pill box carry power that can be calculated from the currents as shown in Fig. 2(b).

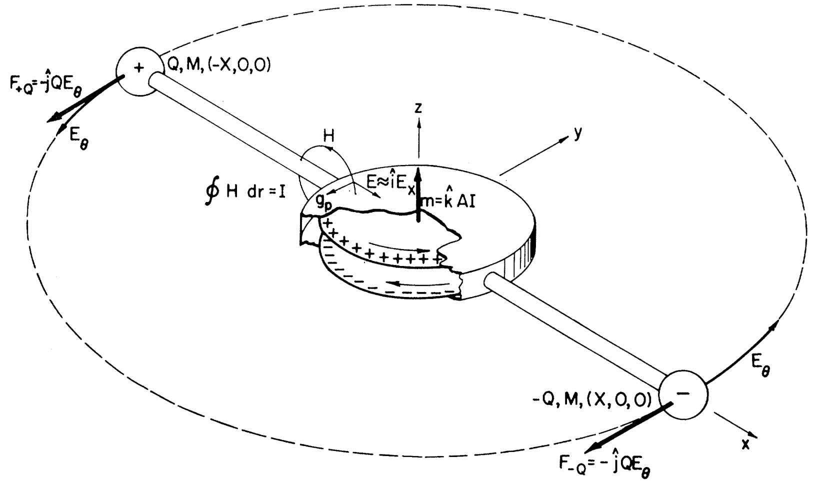

FIG. 1. Conceptual experiment of idealized current loop producing E_θ and transmitting equal linear impulses $G_Q/2$ in $-y$ direction to charges $\pm Q$ as $I \to 0$. Idealized limit of zero space between plates, $\sigma = 0$, $\mu = \mu_0$, and $\epsilon = \epsilon_0$ for all materials.

Since power flow δP over a distance L leads to momentum $\delta PL/c^2$, it is easily concluded by summing the powere $\delta P = -IE\cdot\delta\vec{r}$ of Fig. 2(b) that the momentum G_p is equal to G_Q:

$$G_p = \sum \vec{r}\,\delta P/c^2 = E\times IA/c^2 = \epsilon_0 E\times\mu_0 m = \int g_p dV = -\hat{j}Q\mu_0 IA/2\pi X^2 = G_Q, \tag{8}$$

the integral being obtained from applying Gauss's theorem to $\sum\vec{r}\,\delta P$ with proper regard to sign and using $\nabla\cdot g_p = 0$:

$$(1/c^2)\int\vec{r}dP = \int\vec{r}(g_p\cdot ds) = \int g_p dV. \tag{9}$$

(A generalization of this treatment to quasistationary situations is based on an energy flow $c^2 g_f$ within any closed surface where g_f includes g_p and g_l. One concludes that for specific fields at the surface, the energy-flow momentum

$$G_f = \int g_f dV \tag{10}$$

within the surface is independent of the model used for magnetism and hence that conversion of "hidden momentum" plus G_p within a surface will give the same accelerations to centers of mass regardless of the model used. This is noted below for the specific simple case considered in this Letter, that of magnetic moment changing in the presence of an electric field. It appears that G_f is the "key attribute"[3] in analyzing the conversion of electromagnetic momentum in fields outside magnetic material to ordinary mechanical momentum of the material.)

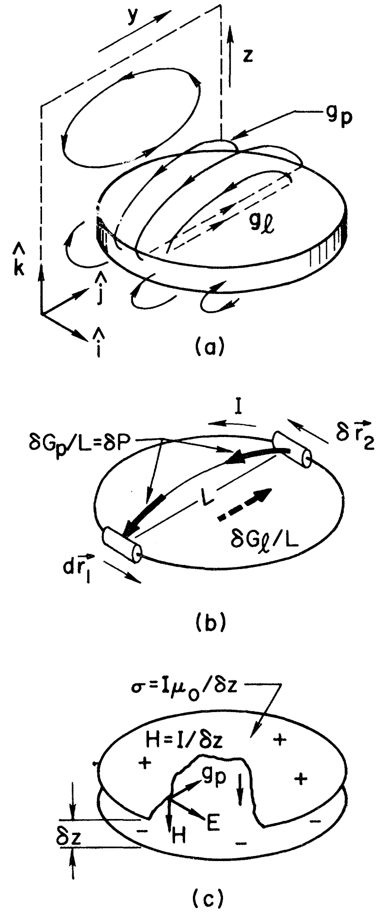

FIG. 2. The origin of the internal momentum G_l for current loop and its equivalence to the integrated Poynting's vector momentum for magnetic shell of equal moment. (a) Distribution of Poynting's vector momentum density g_p and internal momentum g_l for current loop. (b) Calculation of G_p and G_l from the power sources and sinks δP. (c) Equivalence of G_l and G_p for magnetic shell (magnetic charge dipole layer) whose magnetic moment is equal that of current loop.

In order for the center of mass in Fig. 1 to be initially at rest, the divergence of the total energy flow and its associated momentum g must vanish; hence, there must be some form of energy flow with momentum

$$G_l = -G_p \tag{11}$$

within the layer of the pill box. The need for this momentum appears to have been hitherto overlooked.[6,7]

The needed momentum G_l contained in a layer in the pill box would, for the case of m due to the equivalent magnetic shell shown in Fig. 2(c), be given by integration of $E\times H/c^2$ over the volume $A\delta z$ within the shell to give IAE/c^2 as in Eq. (8). For the plastic disk model, G_l is carried by power flow in the form of mechanical stresses and motions in the disks. The mechanical model that, we conjecture, would represent G_l for electronic or nuclear magnetism would produce I by electrically charged inertial masses sliding on circular tracks; G_l would then arise from the higher kinetic energies of masses moving on the lower potential energy parts of their paths. We also conjecture that corresponding features are contained in relativistic quantum mechanics.

Consideration of the electromagnetic stresses over the surface of the pill box shows that no net force is exerted on the pill box as $m = IA$ changes except a negligible one exerted on the vanishingly small cylindrical surfaces. Consequently, momentum is separately conserved both inside and outside the pill box while m and G_l vanish so that

$$G_p \rightarrow G_Q \text{ and } G_l \rightarrow G_D, \tag{12}$$

thus conserving total momentum in Fig. 1 by contributing the needed impulse $G_D = -G_Q$ to the disks. Since the conversion of G_l into G_D can be accomplished only by setting in motion the center of mass of whatever matter carries the current I, the contribution $\epsilon_0 E \times \mu_0 \dot{M}$ to G_D appears as a force acting on the center of mass. Since G_p outside the pill box and consequently G_f inside are uniquely determined by E and m, no matter how m is produced, it follows that the acceleration of the center of mass of the matter in the pill box will always have exactly the value expected for a current of magnetic charges. Similar conclusions based on considerations of G_f can be reached for cases in which E changes.

One experimental prediction of this theory is a quantized deflection of a longitudinally magnetized molecular beam entering a transverse electric field. It also appears to be possible, at least in principle, to detect experimentally the effect of G_l in imparting ordinary mechanical momentum to magnetic material in electromagnetic fields. This cannot be done by simply measuring the average force exerted on magnetic material in ac fields but requires a measurement of the phase of the force with respect to applied fields or other equivalent measurement. Performance of such an experiment here is presently being considered.

The authors appreciate valuable discussions with L. R. Walker and R. L. White. In addition, we appreciate the improvements in presentation resulting from the comments by D. L. Webster and H. A. Haus on the manuscript.

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[1] See, for example, J. A. Stratton, Electromagnetic Theory (McGraw-Hill Book Company, New York, 1941), p. 464.

[2] F. M. Fano, L. J. Chu, and R. B. Adler, Electromagnetic Fields, Energy and Forces (John Wiley & Sons, Inc., New York, 1960), p. 272, suggest such forces with emphasis on electric dipoles produced by currents of magnetic charges. O. Costa de Beauregard, Phys. Letters 24A, 177 (1967), predicts the force $\epsilon_0 \boldsymbol{E} \times \mu_0 \dot{\boldsymbol{M}}$ for current loops by reasoning, similar to other references, based on the action-reaction principle and by assuming equivalence of current loop and magnetic shells. Like other discussions this treatment seems inadequate since the essential role of $\boldsymbol{G}_l$ is not considered.

[3] These "search thinking tools" are discussed in W. Shockley and W. A. Gong, Mechanics (G. E. Merrill, Columbus, Ohio, 1966). See also W. Shockley, IEEE Spectrum 3, 49 (1966).

[4] R. P. Feynman, R. B. Leighton, and M. Sands, The Feynman Lectures on Physics (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1964), Vol. 2, p. 17-5.

[5] For a discussion of $\boldsymbol{D} \times \boldsymbol{B}$ versus $E \times H$ in g_p, see C. Møller, The Theory of Relativity (Oxford University Press, Oxford, England, 1962), p. 205. It is possible that our G_l considerations will affect these questions.

[6] D. L. Webster has considered the need for such momentum, in general, but not in the form of G_l as needed for f_m (personal communication). For examples in which G_l is involved but apparently not considered, including the problem of divergence of $E \times \boldsymbol{H}$ for a magnet in an electric field, see Fano, Chu, and Adler, Ref. 2, p. 312; Feynman, Leighton, and Sands, Ref. 4, p. 27-8; and Costa de Beauregard, Ref. 2.

[7] (Added in proof) H. A. Haus informs us that in a monograph in press he has considered in detail the momentum G_l associated with charged particles moving to produce the current loop and also that he comes to the conclusion that g_p is $E \times \boldsymbol{H}/c^2$ in matter.